

RTT REALIZATION OF QUANTUM AFFINE SUPERALGEBRAS AND TENSOR PRODUCTS

HUAFENG ZHANG

ABSTRACT. We use the RTT realization of the quantum affine superalgebra associated with the Lie superalgebra $\mathfrak{gl}(M, N)$ to study its finite-dimensional representations and their tensor products. In the case $\mathfrak{gl}(1, 1)$, the cyclicity condition of tensor products of finite-dimensional simple modules is determined completely in terms of zeros and poles of rational functions. This in turn induces cyclicity of some particular tensor products of Kirillov-Reshetikhin modules related to $\mathfrak{gl}(M, N)$.

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1. INTRODUCTION

Let q be a non-zero complex number which is not a root of unity. Let $\mathfrak{g} := \mathfrak{gl}(M, N)$ be the *general linear Lie superalgebra*. Let $U_q(\widehat{\mathfrak{g}})$ be the associated quantum affine superalgebra. (We refer to §3.2 for the precise definition.) This is a Hopf superalgebra neither commutative nor co-commutative, and it can be seen as a deformation of the universal enveloping algebra of the following affine Lie superalgebra:

$$L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{1 \leq i, j \leq M+N} E_{ij} \otimes \mathbb{C}[t, t^{-1}].$$

Here the E_{ij} for $1 \leq i, j \leq M + N$ are the elementary matrices in \mathfrak{g} .

In this paper we are mainly concerned with the structure of tensor products of finite-dimensional simple $U_q(\widehat{\mathfrak{g}})$ -modules.

1.1. Backgrounds. Quantum superalgebras appear as the algebraic supersymmetries of some solvable models. For example, the quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}(M, N)})$ is the quantum supersymmetry analogue of the supersymmetric $t - J$ model (with or without a

boundary). A key problem is to diagonalize the commuting transfer matrices. In principle, this can be achieved [Ko13] by constructing the bosonization of vertex operators, which are built over some highest weight Fock representations of $U_q(\widehat{\mathfrak{sl}(M, N)})$.

Another main interest in quantum superalgebras comes from the integrability structure in the context of the AdS/CFT correspondence [Be12]. In this case, the underlying simple Lie superalgebra is $\mathfrak{psl}(2, 2)$, which is the quotient of Lie superalgebra $\mathfrak{sl}(2, 2)$ by its center, the line generated by the identity matrix. A striking feature differing $\mathfrak{psl}(2, 2)$ from all the other simple Lie superalgebras (including simple Lie algebras) is that this simple Lie superalgebra admits a non-trivial three-fold central extension. Based on the Lie superalgebra $\mathfrak{psl}(2, 2)$, several quantum superalgebras have been built as algebraic supersymmetries in AdS/CFT and the closely related Hubbard model: the quantum deformation of extended $\mathfrak{sl}(2, 2)$ in [BK08], the quantum affine deformation of extended $\mathfrak{sl}(2, 2)$ in [BGM12], and the conventional Yangian of extended $\mathfrak{sl}(2, 2)$ in [Be06, BD14], to name a few. Representations of these superalgebras have been considered from different perspectives: [Be07, MM14] for centrally extended $\mathfrak{sl}(2, 2)$ and [ADT10] for the conventional Yangian. For the quantum (affine) superalgebra of extended $\mathfrak{sl}(2, 2)$, only 4-dimensional fundamental representations and R -matrices arising from their tensor products were discussed in [BK08, BGM12].

More closely related to our present paper is the work of Bazhanov-Tsuboi [BT08] on Baxter's \mathbf{Q} -operators related to the quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}(2, 1)})$. In *loc. cit* they constructed the so-called *oscillator representations* of the upper Borel subalgebra \mathfrak{B}_+ . These representations gave rise directly to the \mathbf{Q} -operators and therefore found remarkable applications in spin chain models and in quantum field theory. Their oscillation construction has been generalized to the quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}(M, N)})$ in a recent paper of Tsuboi [Ts12] by using RTT realization.

On the other hand, Hernandez-Jimbo [HJ12] constructed similar oscillator representations of the upper Borel subalgebra \mathfrak{B}_+ of an arbitrary non-twisted quantum affine algebra. In their context, oscillator representations were realized as certain asymptotic limits of Kirillov-Reshetikhin modules over the quantum affine algebra, hence bearing the name *asymptotic representations*. The asymptotic construction enabled Frenkel-Hernandez [FH13] to give a representation theoretic interpretation of Baxter's \mathbf{T} - \mathbf{Q} relations and to solve a conjecture of Frenkel-Reshetikhin on the spectra of quantum integrable systems [FR99].

Based on the above progress, it is natural to consider representation theory of the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$, and more specifically the quantum superalgebras related to centrally extended $\mathfrak{sl}(2, 2)$. In the present paper, $U_q(\widehat{\mathfrak{g}})$ is our main concern.

We are motivated by the following question: can the oscillator representations related to the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ in [BT08, Ts12] be realized as asymptotic limits of Kirillov-Reshetikhin modules in the spirit of Hernandez-Jimbo?

In the super case, the representation theory of quantum affine superalgebras is still less developed, compared to the vast literature on representations of quantum affine algebras (see the two review papers [CH10, Le11]).

1.2. Representations of $U_q(\widehat{\mathfrak{g}})$. In a recent paper [Zh13], we obtained a classification of finite-dimensional simple modules for the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$. For the Kac-Moody superalgebra \mathfrak{g} , let $I_0 := \{1, 2, \dots, M + N - 1\}$ be the set of vertices of the

distinguished Dynkin diagram. Hence $i \in I_0$ corresponds to the simple root α_i and α_M is an odd isotopic simple root. The main result in *loc. cit* can be stated as follows: up to tensor product by one-dimensional modules, finite-dimensional simple $U_q(\widehat{\mathfrak{g}})$ -modules are of the form $S(\underline{f})$ where $\underline{f} = (f_i)_{i \in I_0}$ is an I_0 -tuple of rational functions $f_i(z) \in \mathbb{C}(z)$ such that:

- (a) if $i \neq M$ then there exists a polynomial $P_i(z)$ with constant term 1 such that $f_i(z) = q_i^{\deg P_i} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}$. Here $q_i = q$ for $i \leq M$ and q^{-1} otherwise;
- (b) if $i = M$, then $f_i(z)$ as a meromorphic function is regular at $z = 0$ and $z = \infty$. Moreover, $f_i(0)f_i(\infty) = 1$.

We remark that (a) implies (b) but not vice versa. Hence this classification result is different from the case of quantum affine algebras [CP91].

In analogy with the non-graded case, Kirillov-Reshetikhin modules for $U_q(\widehat{\mathfrak{g}})$ will be those modules $S(\varpi_{n,a}^{(i)})$ where $i \in I_0$ is a Dynkin vertex, $a \in \mathbb{C}^\times$ is a spectral parameter, $n \in \mathbb{Z}_{>0}$ is a positive integer, and $\varpi_{n,a}^{(i)}$ is the I_0 -tuple of rational functions whose i -th coordinate is $q_i^n \frac{1-zaq_i^{-n}}{1-zaq_i^n}$ and whose other coordinates are 1. When $n = 1$, the Kirillov-Reshetikhin modules are also called *fundamental modules*.

1.2.1. Asymptotic limits. Let us fix a Dynkin vertex $i \in I_0$ and a spectral parameter $a \in \mathbb{C}^\times$. For $n \in \mathbb{Z}_{>0}$, the i -th coordinate for $\varpi_{n,aq_i^n}^{(i)}$ has the asymptotic expression $q_i^n \frac{1-za}{1-zaq_i^{2n}}$.

Informally, by taking asymptotic limit of the Kirillov-Reshetikhin modules $S(\varpi_{n,aq_i^n}^{(i)})$ we should get a “module” where the i -th coordinate is $1 - za$ (by first forgetting the constant term q_i^n and then taking the analysis limit $\lim_{n \rightarrow \infty} q_i^n = 0$). This module should be an oscillator module.

The above intuitive argument was made mathematically rigorous in [HJ12], where inductive/projective systems of Kirillov-Reshetikhin modules were constructed and their inductive/projective limits were shown to be oscillator modules. One of the main ingredients for the construction of these systems is a cyclicity property of tensor products of Kirillov-Reshetikhin modules of a particular form. Also a result of [He10, Proposition 3.2] on q -characters of tensor products of simple modules was needed to establish stability and asymptotic properties of these systems.

1.2.2. Tensor products of Kirillov-Reshetikhin modules. Let us explain in detail the cyclicity result used in [HJ12]. In this paragraph let us replace the Lie superalgebra \mathfrak{g} by an arbitrary simple Lie algebra \mathfrak{g}'' . The set J of Dynkin vertices, the numbers q_j for $j \in J$, and the Kirillov-Reshetikhin modules $S(\varpi_{n,a}^{(j)})$ are similarly defined. Then

- (C) the tensor products $\bigotimes_{l=1}^k S(\varpi_{1,aq_j^{-2l}}^{(j)})$ for $k \geq 1$ are cyclic.

Here being *cyclic* means being of highest ℓ -weight with respect to the Drinfeld type triangular decomposition of $U_q(\widehat{\mathfrak{g}}'')$.

Let us give a quick overview of cyclicity property of tensor products of finite-dimensional simple modules over $U_q(\widehat{\mathfrak{g}}'')$. We refer to [CH10, §5] for more historical comments. In [AK97, Conjecture 1] it was conjectured by Akasaka-Kashiwara that

(AK) let V_1, V_2, \dots, V_n be fundamental $U_q(\widehat{\mathfrak{g}}'')$ -modules and let $x_1, x_2, \dots, x_n \in \mathbb{Z}$. Then the tensor product $\bigotimes_{i=1}^n (V_i)_{q^{x_i}}$ is cyclic if $x_i \geq x_j$ for all $1 \leq i < j \leq n$.

This conjecture has been proved in the case of type $A_n^{(1)}$ and $C_n^{(1)}$ in *loc. cit* and later by Kashiwara [Ka02, Theorem 9.1] in full generality. Both proofs relied on crystal bases theory for modules over the quantum affine algebra. Now (C) is a direct consequence of (AK).

At the same time, a geometric proof of (AK) in the simply laced case was given by Varagnolo-Vasserot using quiver varieties [VV02, Corollary 7.17].

Also (AK) was generalized by Chari [Ch02, Theorem 4.4] with a more Lie theoretic and algebraic approach. By using the Weyl group action on the set of weights of a $U_q(\mathfrak{g}'')$ -module, and the Braid group action on the affine Cartan subalgebra of $U_q(\widehat{\mathfrak{g}}'')$, Chari reduced the cyclicity problem on $U_q(\widehat{\mathfrak{g}}'')$ -modules to a series of similar problems on $U_q(\widehat{\mathfrak{sl}}_2)$ -modules corresponding to a fixed reduced expression of the longest element w_0 in the Weyl group. Eventually a sufficient condition for a tensor products of simple modules $S(\underline{f})$ to be cyclic was given in terms of Drinfeld polynomials defining these \underline{f} as in (a).

1.2.3. The super case. To construct asymptotic limits, we need inevitably such cyclicity property as (C) of tensor products of Kirillov-Reshetikhin modules over the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$. However, the main techniques used in the non-graded case to deduce cyclicity results cannot be applied directly to the super case. For example, crystal base theory and quiver geometry for quantum affine superalgebras, or even for finite type quantum superalgebras, are still less developed [BKK00]. The main drawback comes from the fact that the Weyl group of \mathfrak{g} , being $\mathfrak{S}_M \times \mathfrak{S}_N$ instead of \mathfrak{S}_{M+N} , is too small to capture enough information on weights and linkage.

Nevertheless, we can prove a weak version of (AK) (yet stronger than (C)) for quantum affine superalgebras, by modifying the arguments of Chari in [Ch02]. Although our motivation of studying the cyclicity property of tensor products comes from the asymptotic construction, we think that cyclicity property is of independent interest, and a large part of the present paper is devoted to proving this weak version of (AK), Theorem 4.2.

For this purpose, we shall adopt the RTT realization of the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ instead of the Drinfeld realization. The main advantages are that: first of all, RTT generators are quantum analogues of such loop generators $E_{ij} \otimes t^n \in L\mathfrak{g}$; secondly and more importantly, RTT generators have nice coproduct formulas. Our present work is inspired on the one hand by the work [MTZ04] of Molev, Tolstoy and Rui-Bin Zhang on simplicity of tensor products of evaluation modules for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$, where RTT realization and coproduct formulas for RTT generators made the relevant calculations transparent, and on the other hand by the work of Tsuboi [Ts12] on oscillation constructions using RTT realization as mentioned before.

In comparison to the non-graded case [AK97, Ch02, Ka02, VV02], our approach of studying cyclicity of tensor products differs from the perspective that we use (quantum analogue of) root vectors of the quantum affine superalgebra instead of Weyl groups. This is an idea already explored in our previous paper [Zh13] on classification of finite-dimensional simple modules, where Weyl group was replaced by Poincaré-Birkhoff-Witt linear generators in terms of Drinfeld generators for the quantum affine superalgebra. In particular, our approach applies also to quantum affine algebras of type A (non-graded).

1.3. Main results. We study in full detail the RTT realization of the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$, including its definition, different kinds of grading, its degeneration to the finite type quantum superalgebra $U_q(\mathfrak{g})$, evaluation morphisms, its relationship with quantum double construction, and coproduct formulas for Drinfeld generators. Almost all the relevant results are proved in a uniform way. This makes the present paper longer than we have expected.

The first main result is an analogue of (AK) under the assumption that the fundamental modules V_i are the same (Theorem 4.2).

The idea of proof follows largely that of Chari [Ch02]. The RTT generators will replace the role of the Weyl group. The quantum analogues of $E_{1,M+N} \otimes t^n, E_{M+N,1} \otimes t^n$ will be candidates for the longest element w_0 in the Weyl group. For the reduction argument, we will use representation theory of the q -Yangian $Y_q(\mathfrak{gl}(1,1))$ instead of $U_q(\widehat{\mathfrak{sl}}_2)$. Here q -Yangian $Y_q(\mathfrak{g})$ is a sub-Hopf-superalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by half of the RTT generators. It can be viewed as the upper Borel subalgebra.

Our second main result (Theorem 5.2) is on representation theory of $Y_q(\mathfrak{gl}(1,1))$.

- (1) There is a classification of finite-dimensional simple modules, up to tensor product by one-dimensional modules, in terms of highest ℓ -weights parametrized by the set \mathbf{R} of such rational functions $f(z)$ that $f(0) = 1$ (hence regular at $z = 0$). Let $V(f)$ be the simple module of highest ℓ -weight f .
- (2) For $f_1, \dots, f_k \in \mathbf{R}$, the tensor product $\bigotimes_{j=1}^k V(f_j)$ is of highest ℓ -weight (resp. of lowest ℓ -weight) if and only if: for all $1 \leq i < j \leq k$ (resp. for all $1 \leq j < i \leq k$) the set of poles of f_i does not intersect with the set of zeros of f_j .
- (3) The tensor product in (2) is simple if and only if it is of highest and lowest ℓ -weight.

We can see (2) as a stronger improvement of (AK) for the q -Yangian $Y_q(\mathfrak{gl}(1,1))$ as the necessary condition of cyclicity is also described. However, (2) cannot be generalized to higher rank quantum affine superalgebras or q -Yangians. Indeed, (2) already fails if we replace $Y_q(\mathfrak{gl}(1,1))$ by the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, as seen in [CP91, MY14] where the condition for a tensor product of simple $U_q(\widehat{\mathfrak{sl}}_2)$ -modules to be cyclic was more sophisticated. Also, in the non-graded case due to the Weyl group action (more precisely the element w_0) a tensor product of simple modules is of highest ℓ -weight if and only if it is of lowest ℓ -weight. Hence (3) is really a special feature in the super case.

Except Chari's Lemma which requires coproduct formulas of Drinfeld generators, the proof of Theorem 5.2 is quite elementary and explicit. Eventually we arrive at a classical linear algebra problem on determining linear independence of some polynomials of a particular form (Lemma 5.4).

Surprisingly, reductions from $U_q(\widehat{\mathfrak{g}})$ -modules to $Y_q(\mathfrak{gl}(1,1))$ -modules are already enough to prove Theorem 4.2. We believe that more general cyclicity results can be deduced in this way, although this needs extra efforts. See the end of §6.

In an upcoming paper [Zh], we shall use Theorem 4.2 to construct inductive systems of Kirillov-Reshetikhin modules and realize their limits as asymptotic modules over the q -Yangian $Y_q(\mathfrak{g})$, hence extending Hernandez-Jimbo's asymptotic construction to the super case. Indeed, the $Y_q(\mathfrak{gl}(1,1))$ -modules $V(1-z)$ and $V(\frac{1}{1-z})$ can be viewed as prototypes of asymptotic modules.

At last, we would like to point out that nearly all the results in the present paper have direct analogues when replacing the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ (or the q -Yangian) by the Yangian $Y(\mathfrak{g})$, a deformation of the universal enveloping algebra of the current Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t]$. The proofs of these results are essentially the same, as $Y(\mathfrak{g})$ admits a similar RTT realization [Zr96]. In [Zr95, Theorem 5], a similar criteria for a tensor product of finite-dimensional simple $Y(\mathfrak{gl}(1, 1))$ -modules to be simple was given by Rui-Bin Zhang with a quite different approach from ours. Cyclicity of tensor products and Drinfeld realization for the Yangian were not considered there in full generality.

1.4. Outline. The paper is organized as follows. §2 collects some basic facts about highest weight representations of the finite type quantum superalgebra $U_q(\mathfrak{g})$. In §3 we study in detail the RTT realization of the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$, review the Ding-Frenkel homomorphism between Drinfeld realization and RTT realization, and give an estimation for coproduct of Drinfeld generators (Proposition 3.13), which is needed to prove Chari's lemma in the super case (Lemma 4.5). §4 studies highest ℓ -weight representations for $U_q(\widehat{\mathfrak{g}})$ and states the first main result (Theorem 4.2) on tensor products of Kirillov-Reshetikhin modules. §5 discusses finite-dimensional representation theory for the q -Yangian $Y_q(\mathfrak{gl}(1, 1))$, which is used in §6 to conclude the proof of Theorem 4.2. In §A, we give the complete proof of Proposition 3.13 on coproduct of Drinfeld generators.

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2. PRELIMINARIES

In this section, we introduce the basic notations concerning the quantum superalgebra $U_q(\mathfrak{gl}(M, N))$ and its representations. Following Benkart-Kang- Kashiwara we review the character formula for fundamental representations.

2.1. Conventions. Throughout this paper, q is fixed to be a non-zero complex number and not a root of unity. All the vector superspaces and superalgebras are defined over the base field \mathbb{C} . Recall that a vector superspace V is a vector space V with a \mathbb{Z}_2 -grading $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We write $|x| = i$ for $i \in \mathbb{Z}_2$ and $x \in V_i$. It will happen usually that V carries another grading by an abelian group P . In this case, we write $V = \bigoplus_{\alpha \in P} (V)_{\alpha}$ (we keep the parenthesis most of the time) and $|x|_P = \alpha$ for $\alpha \in P$ and $x \in (V)_{\alpha}$.

Fix $M, N \in \mathbb{Z}_{\geq 0}$. Set $I := \{1, 2, \dots, M + N\}$. Define two maps as follows:

$$|\cdot| : I \longrightarrow \mathbb{Z}_2, i \mapsto |i| := \begin{cases} \bar{0} & (i \leq M), \\ \bar{1} & (i > M); \end{cases} \quad d : I \longrightarrow \mathbb{Z}, i \mapsto d_i := \begin{cases} 1 & (i \leq M), \\ -1 & (i > M). \end{cases}$$

Set $q_i := q^{d_i}$. Set $\mathbf{P} := \bigoplus_{i \in I} \mathbb{Z}\epsilon_i$. Let $(,) : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbb{Z}$ be the bilinear form defined by $(\epsilon_i, \epsilon_j) = \delta_{ij}d_i$. Let $|\cdot| : \mathbf{P} \longrightarrow \mathbb{Z}_2$ be the morphism of abelian groups such that $|\epsilon_i| = |i|$.

In the following, *only* three cases of $|x| \in \mathbb{Z}_2$ are admitted: $x \in I$; $x \in \mathbf{P}$; x is a \mathbb{Z}_2 -homogeneous vector of a vector superspace.

Unless otherwise stated, \mathfrak{g} will always be the Lie superalgebra $\mathfrak{gl}(M, N)$, which is, the vector space $\bigoplus_{i,j \in I} \mathbb{C}E_{ij}$ with \mathbb{Z}_2 -grading and Lie bracket:

$$|E_{ij}| = |i| + |j|, \quad [E_{ij}, E_{kl}] = \delta_{jk}E_{il} - (-1)^{(|i|+|j|)(|k|+|l|)}\delta_{il}E_{kj}.$$

Here, we view \mathbb{Z}_2 as a ring and $(-1)^\cdot : \mathbb{Z}_2 \longrightarrow \{1, -1\}$ as the sign map. Let $\mathfrak{h} = \bigoplus_{i \in I} \mathbb{C}E_{ii}$. Then \mathfrak{h} is a Cartan subalgebra with respect to which \mathfrak{g} has a root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{i,j \in I, i \neq j} (\mathfrak{g})_{\epsilon_i - \epsilon_j} \right), \quad (\mathfrak{g})_{\epsilon_i - \epsilon_j} = \{x \in \mathfrak{g} \mid [E_{kk}, x] = (\delta_{ik} - \delta_{jk})x \text{ for } k \in I\} = \mathbb{C}E_{ij}.$$

Set $I_0 := I \setminus \{M + N\}$. For $i \in I_0$, set $\alpha_i := \epsilon_i - \epsilon_{i+1} \in \mathbf{P}$. Introduce the root lattice $\mathbf{Q} := \bigoplus_{i \in I_0} \mathbb{Z}\alpha_i \subset \mathbf{P}$. Define $\mathbf{Q}_{\geq 0} := \bigoplus_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$.

2.2. The quantum superalgebra $U_q(\mathfrak{g})$. This is the superalgebra defined by generators $e_i^\pm, t_j^{\pm 1}$ ($i \in I_0, j \in I$) with \mathbb{Z}_2 -degrees $|e_i^\pm| = |\alpha_i|$ and $|t_j^{\pm 1}| = 0$ and with relations

$$\begin{aligned} t_j t_j^{-1} &= t_j^{-1} t_j = 1, \quad t_i t_j = t_j t_i \quad (i, j \in I), \\ t_i e_j^\pm t_i^{-1} &= q^{\pm d_i(\epsilon_i, \epsilon_j - \epsilon_{j+1})} e_j^\pm \quad (i \in I, j \in I_0), \\ [e_i^+, e_j^-] &= \delta_{ij} \frac{t_i^{d_i} t_{i+1}^{-d_{i+1}} - t_i^{-d_i} t_{i+1}^{d_{i+1}}}{q_i - q_i^{-1}} \quad (i, j \in I_0), \end{aligned}$$

together with Serre relations which we do not repeat (see [Zh13, §2.2] for example). $U_q(\mathfrak{g})$ has a Hopf superalgebra structure with coproduct: for $i \in I_0, j \in I$

$$\Delta(e_i^+) = 1 \otimes e_i^+ + e_i^+ \otimes t_i^{-d_i} t_{i+1}^{d_{i+1}}, \quad \Delta(e_i^-) = t_i^{d_i} t_{i+1}^{-d_{i+1}} \otimes e_i^- + e_i^- \otimes 1, \quad \Delta(t_j) = t_j \otimes t_j.$$

There exists a \mathbf{Q} -grading on $U_q(\mathfrak{g})$ respecting the Hopf superalgebra structure:

$$|t_j|_{\mathbf{Q}} = 0, \quad |e_i^\pm|_{\mathbf{Q}} = \pm \alpha_i \quad (i \in I_0, j \in I).$$

In general, for $\alpha \in \mathbf{Q} \subset \mathbf{P}$, we have

$$(U_q(\mathfrak{g}))_\alpha = \{x \in U_q(\mathfrak{g}) \mid t_i x t_i^{-1} = q^{d_i(\epsilon_i, \alpha)} x \text{ for } i \in I\}.$$

This \mathbf{Q} -grading is compatible with the \mathbb{Z}_2 -grading: $(U_q(\mathfrak{g}))_\alpha \subset U_q(\mathfrak{g})_{|\alpha|}$ for $\alpha \in \mathbf{Q}$.

2.3. Highest weight representations. Let $\lambda \in \mathbf{P}$. Up to isomorphism, there exists a unique simple $U_q(\mathfrak{g})$ -module, denoted by $L(\lambda)$, which is generated by a vector v_λ satisfying:

$$|v_\lambda| = |\lambda|, \quad e_i^+ v_\lambda = 0, \quad t_j v_\lambda = q^{d_j(\epsilon_j, \lambda)} v_\lambda \quad (i \in I_0, j \in I).$$

The action of the t_j endows $L(\lambda)$ with the following \mathbf{P} -grading:

$$(L(\lambda))_\alpha := \{x \in L(\lambda) \mid t_j x = q^{d_j(\epsilon_j, \lambda)} x \text{ for } j \in I\}.$$

Using the triangular decomposition for $U_q(\mathfrak{g})$, one can show the following: $(L(\lambda))_\lambda = \mathbb{C}v_\lambda$; the \mathbf{P} -grading on $L(\lambda)$ is compatible with the \mathbb{Z}_2 -grading; for $\alpha \in \mathbf{P}$, $(L(\lambda))_\alpha \neq 0$ only if $\lambda - \alpha \in \mathbf{Q}_{\geq 0}$.

It was shown [Zr93] that for $\lambda \in \mathbf{P}$, $L(\lambda)$ is finite-dimensional if and only if $d_i(\lambda, \alpha_i) > 0$ for all $i \in I_0 \setminus \{M\}$.

Example 1. (Natural representation.) Let $\mathbf{V} = \oplus_{i \in I} \mathbb{C}v_i$ be the vector superspace with \mathbb{Z}_2 -grading $|v_i| = |i|$. On \mathbf{V} there is a natural representation $\rho_{(0)}$ of $U_q(\mathfrak{g})$ defined by:

$$\rho_{(0)}(e_i^+) = E_{i,i+1}, \quad \rho_{(0)}(e_i^-) = E_{i+1,i}, \quad \rho_{(0)}(t_j) = \sum_{k \in I} q^{d_j(\epsilon_j, \epsilon_k)} E_{kk} \quad (i \in I_0, j \in I).$$

Here the $E_{ij} \in \text{End}(\mathbf{V})$ for $i, j \in I$ are defined by $E_{ij}v_k = \delta_{jk}v_i$. Clearly, $\mathbf{V} = L(\epsilon_1)$ as a $U_q(\mathfrak{g})$ -module with v_1 a highest weight vector, and $(\mathbf{V})_{\epsilon_i} = \mathbb{C}v_i$ for $i \in I$.

Example 2. Consider the tensor product $\mathbf{V}^{\otimes 2}$ as a $U_q(\mathfrak{g})$ -module. Define subspaces

$$(2.1) \quad \mathbf{V}^+ := \bigoplus_{1 \leq i < j \leq M+N} \mathbb{C}(qv_i \otimes v_j + (-1)^{|i||j|}v_j \otimes v_i) \oplus \bigoplus_{k=1}^M \mathbb{C}(v_k \otimes v_k),$$

$$(2.2) \quad \mathbf{V}^- := \bigoplus_{1 \leq i < j \leq M+N} \mathbb{C}(q^{-1}v_i \otimes v_j - (-1)^{|i||j|}v_j \otimes v_i) \oplus \bigoplus_{k=1}^N \mathbb{C}(v_{M+k} \otimes v_{M+k}).$$

Then $\mathbf{V}^{\otimes 2} = \mathbf{V}^+ \oplus \mathbf{V}^-$ is a decomposition of $U_q(\mathfrak{g})$ -modules as follows:

$$L(\epsilon_1)^{\otimes 2} = L(2\epsilon_1) \oplus L(\epsilon_1 + \epsilon_2).$$

In the following, the three vector superspaces $\mathbf{V}, \mathbf{V}^+, \mathbf{V}^-$ will be used frequently.

2.3.1. Characters. Let V be a $U_q(\mathfrak{g})$ -modules \mathbf{P} -graded via the action of the t_i :

$$(V)_\alpha = \{x \in V \mid t_i x = q^{d_i(\alpha, \epsilon_i)} x \text{ for } i \in I\}.$$

Suppose that all the weight spaces $(V)_\alpha$ are finite-dimensional. Introduce *characters*

$$(2.3) \quad \chi(V) := \sum_{\alpha \in \mathbf{P}} \dim(V)_\alpha [\alpha] \in \mathbb{Z}^{\mathbf{P}}.$$

Here $\mathbb{Z}^{\mathbf{P}}$ is the abelian group of functions $\mathbf{P} \rightarrow \mathbb{Z}$ and $[\alpha] = \delta_{\alpha, \cdot}$. By definition, it is clear that $(U_q(\mathfrak{g}))_\alpha (V)_\beta \subseteq (V)_{\alpha+\beta}$ for $\alpha, \beta \in \mathbf{P}$.

2.3.2. Fundamental weights. For $r \in I_0$, let us define the r -th *fundamental weight*

$$\varpi_r := \begin{cases} \sum_{i=1}^r \epsilon_i & (r \leq M), \\ -\sum_{i=r+1}^{M+N} \epsilon_i & (r > M). \end{cases}$$

Clearly, $d_s(\varpi_r, \alpha_s) = \delta_{rs}$ for $r, s \in I_0$.

For $1 \leq r \leq M$ let \mathcal{B}_r be the set of functions $f : \{1, 2, \dots, r\} \rightarrow I$ such that: $f(i) \leq f(i')$ for $i \leq i'$; if $1 \leq i < r$ and $f(i) \leq M$, then $f(i) < f(i+1)$.

Theorem 2.1. [BKK00] *For $1 \leq r \leq M$ we have*

$$(2.4) \quad \chi(L(\varpi_r)) = \sum_{f \in \mathcal{B}_r} \left[\sum_{i=1}^r \epsilon_{f(i)} \right] \in \mathbb{Z}[\mathbf{P}].$$

□

Similar character formula for the simple modules $L(\varpi_r)$ with $M < r < M + N$ can be obtained with the help of a superalgebra isomorphism $U_q(\mathfrak{gl}(M, N)) \longrightarrow U_q(\mathfrak{gl}(N, M))$. We shall return to this point later (Remark 4.4).

3. QUANTUM AFFINE SUPERALGEBRA AND q -YANGIANS

In this section, we introduce our central objects of study: the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ and the q -Yangian $Y_q(\mathfrak{g})$ within the framework of RTT. Most of the results in this section have appeared in the literature separately (for example [FM02, MTZ04, Ts12]). For completeness and for later reference, we prove all of them in a uniform way, except the Ding-Frenkel homomorphism relating Drinfeld realization and RTT realization, which requires lengthy calculations as done in [DF93, Zy97].

3.1. Yang-Baxter algebras. We say that $R \in \text{End}(\mathbf{V} \otimes \mathbf{V})$ is an R -matrix if: R is invertible and of \mathbb{Z}_2 -degree $\bar{0}$; R satisfies the *Yang-Baxter equation*

$$(3.5) \quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in (\text{End}\mathbf{V})^{\otimes 3}.$$

Here $R_{12} = R \otimes \text{Id}_{\mathbf{V}}$, $R_{23} = \text{Id}_{\mathbf{V}} \otimes R$ and

$$R_{13} = (\text{Id}_{\mathbf{V}} \otimes c_{\mathbf{V}, \mathbf{V}})R_{12}(\text{Id}_{\mathbf{V}} \otimes c_{\mathbf{V}, \mathbf{V}}) = (c_{\mathbf{V}, \mathbf{V}} \otimes \text{Id}_{\mathbf{V}})R_{23}(c_{\mathbf{V}, \mathbf{V}} \otimes \text{Id}_{\mathbf{V}})$$

with $c_{\mathbf{V}, \mathbf{V}} : \mathbf{V} \otimes \mathbf{V} \mapsto \mathbf{V} \otimes \mathbf{V}$, $v_i \otimes v_j \mapsto (-1)^{|i||j|} v_j \otimes v_i$ the braiding.

Definition 3.1. For an R -matrix $R \in \text{End}(\mathbf{V} \otimes \mathbf{V})$, the *Yang-Baxter algebra* $\mathcal{YB}(R)$ is the superalgebra with

(YB1) generators l_{ij} for $i, j \in I$ of \mathbb{Z}_2 -degrees $|l_{ij}| = |i| + |j|$;

(YB2) relations $R_{23}L_{12}L_{13} = L_{13}L_{12}R_{23} \in \mathcal{YB}(R) \otimes \text{End}\mathbf{V} \otimes \text{End}\mathbf{V}$.

Here $L = \sum_{i,j \in I} l_{ij} \otimes E_{ij} \in \mathcal{YB}(R) \otimes \text{End}\mathbf{V}$.

Already from the definition of R -matrix, one reads a natural representation of $\mathcal{YB}(R)$.

Lemma 3.2. There is a representation (ρ, \mathbf{V}) of the superalgebra $\mathcal{YB}(R)$ on \mathbf{V} defined by $(\rho \otimes \text{Id}_{\text{End}\mathbf{V}})(L) = R^{-1}$.

Proof. This is obvious in view of the following: $S \mapsto S_{23}$ is a morphism of superalgebras $(\text{End}\mathbf{V})^{\otimes 2} \longrightarrow (\text{End}\mathbf{V})^{\otimes 3}$; R^{-1} also satisfies the Yang-Baxter equation. \square

3.1.1. Super bialgebra structure. The Yang-Baxter algebra $\mathcal{YB}(R)$ can be made into a super bialgebra with coproduct and counit

$$(3.6) \quad \Delta(l_{ij}) = \sum_{k \in I} (-1)^{(|i|+|k|)(|j|+|k|)} l_{ik} \otimes l_{kj}, \quad \varepsilon(l_{ij}) = \delta_{ij}.$$

To prove that Δ is a well-defined superalgebra morphism, introduce

$$T := \sum_{i,j,k \in I} (-1)^{(|i|+|k|)(|j|+|k|)} l_{ik} \otimes l_{kj} \otimes E_{ij} = L_{13}L_{23} \in \mathcal{YB}(R) \otimes \mathcal{YB}(R) \otimes \text{End}\mathbf{V}.$$

It is enough to ensure that

$$R_{34}T_{123}T_{124} = T_{124}T_{123}R_{34} \in \mathcal{YB}(R)^{\otimes 2} \otimes (\text{End}\mathbf{V})^{\otimes 2}.$$

Observe first of all that $|L| = \bar{0}$ implies

$$L_{23}L_{14} = L_{14}L_{23}, \quad L_{13}L_{24} = L_{24}L_{13}.$$

It follows that

$$\begin{aligned} R_{34}T_{123}T_{124} &= R_{34}L_{13}(L_{23}L_{14})L_{24} = (R_{34}L_{13}L_{14})L_{23}L_{24} \\ &\stackrel{YB}{=} L_{14}L_{13}(R_{34}L_{23}L_{24}) \stackrel{YB}{=} L_{14}(L_{13}L_{24})L_{23}R_{34} \\ &= L_{14}L_{24}L_{13}L_{23}R_{34} = T_{124}T_{123}R_{34}. \end{aligned}$$

The co-associativity of Δ is clear from Equation (3.6).

Remark that in the above proof we do not need the Yang-Baxter equation for R .

3.1.2. Dependence on R -matrices. Given an R -matrix R , we will construct two new R -matrices R', R'' whose associated Yang-Baxter algebras are isomorphic.

The operator R' is easy to define:

$$R' := c_{\mathbf{V}, \mathbf{V}} R^{-1} c_{\mathbf{V}, \mathbf{V}} \in \text{End}(\mathbf{V} \otimes \mathbf{V}).$$

Before giving the second operator R'' , let us introduce a super version of transpose of matrices. This is the linear map $\tau : \text{End} \mathbf{V} \rightarrow \text{End} \mathbf{V}$ given by

$$(3.7) \quad \tau(E_{ij}) := \varepsilon_{ij} E_{ji}, \quad \varepsilon_{ij} := (-1)^{|i|(|i|+|j|)} = \begin{cases} 1 & (i \leq j), \\ (-1)^{|i|+|j|} & (i > j). \end{cases}$$

Now R'' is given by

$$R'' := (\tau \otimes \tau)(R^{-1}) \in \text{End}(\mathbf{V} \otimes \mathbf{V}).$$

Lemma 3.3. $\tau : \text{End} \mathbf{V} \rightarrow \text{End} \mathbf{V}$ is an anti-automorphism of superalgebra.

Proof. We explain that τ is indeed a duality map. Let \mathbf{V}^* be the dual of \mathbf{V} , with $(v_i^* : i \in I)$ the dual basis with respect to $(v_i : i \in I)$. Then $|v_i^*| = |v_i| = |i|$. Let $e_{ij} \in \text{End} \mathbf{V}^*$ be $v_k^* \mapsto \delta_{jk} v_i^*$. Then the linear isomorphism of vector superspaces $\mathbf{V} \rightarrow \mathbf{V}^*$, $v_i \mapsto v_i^*$ induces an isomorphism of superalgebras $a : \text{End} \mathbf{V} \rightarrow \text{End} \mathbf{V}^*$, $E_{ij} \mapsto e_{ij}$.

On the other hand, let $f : \mathbf{V} \rightarrow \mathbf{V}$ be a \mathbb{Z}_2 -homogeneous linear map. Define its *dual*:

$$f^* : \mathbf{V}^* \rightarrow \mathbf{V}^*, \quad l \mapsto (x \mapsto (-1)^{|l||f|} f(x))$$

where $l \in \mathbf{V}^*$ is \mathbb{Z}_2 -homogeneous and $x \in \mathbf{V}$. In this way, we construct an anti-isomorphism of superalgebras $b : \text{End} \mathbf{V} \rightarrow \text{End} \mathbf{V}^*$, $f \mapsto f^*$. It is straightforward to check that $\tau = a^{-1} \circ b$. Hence τ is an anti-automorphism of superalgebra. \square

Now we can state the main result of this paragraph.

Proposition 3.4. Let $R \in \text{End}(\mathbf{V} \otimes \mathbf{V})$ be an R -matrix. Then R', R'' are also R -matrices. Moreover: the assignment $l'_{ij} \mapsto l_{ij}$ extends to an isomorphism $\mathcal{YB}(R') \rightarrow \mathcal{YB}(R)$ of super bialgebras; the assignment $l''_{ij} \mapsto \varepsilon_{ji} l_{ji}$ extends to an isomorphism $\Psi : \mathcal{YB}(R'') \rightarrow \mathcal{YB}(R)^{\text{cop}}$ of super bialgebras.

Proof. We only prove the part for R'' . To show that R'' is an R -matrix, note that $\tau^{\otimes 2}, \tau^{\otimes 3}$ are anti-automorphisms of superalgebras, from which the Yang-Baxter equation for R'' follows. Next, in order to show that the superalgebra morphism Ψ is well defined, we only need to ensure that

$$R''_{23}T_{12}T_{13} = T_{13}T_{12}R''_{23} \in \mathcal{YB}(R) \otimes \text{End}\mathbf{V} \otimes \text{End}\mathbf{V}$$

where $T = (\text{Id}_{\mathcal{YB}(R)} \otimes \tau)(L) \in \mathcal{YB}(R) \otimes \text{End}\mathbf{V}$.

By applying $\text{Id}_{\mathcal{YB}(R)} \otimes \tau \otimes \tau$ to the Yang-Baxter equation $R_{23}L_{12}L_{13} = L_{13}L_{12}R_{23}$ we get

$$T_{12}T_{13}((\tau \otimes \tau)(R))_{23} = ((\tau \otimes \tau)(R))_{23}T_{13}T_{12},$$

from which comes the desired equation. The rest is clear. \square

Remark that the two operations $R \mapsto R', R \mapsto R''$ are involutions. We will sometimes be in the situation $R' = R''$. Proposition 3.4 then gives us an isomorphism $\Psi : \mathcal{YB}(R) \rightarrow \mathcal{YB}(R)^{\text{cop}}$ of super bialgebras.

3.2. The quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$. Recall the vector superspaces $\mathbf{V}^+, \mathbf{V}^-$ in Example 2. Define the *Perk-Schultz matrix* [PS81]

$$(3.8) \quad R(z, w) = c_{\mathbf{V}, \mathbf{V}}((zq - wq^{-1})P^+ + (wq - zq^{-1})P^-) \in \text{End}(\mathbf{V} \otimes \mathbf{V})[z, w]$$

where P^+, P^- are projections with respect to the decomposition $\mathbf{V}^{\otimes 2} = \mathbf{V}^+ \oplus \mathbf{V}^-$.

Definition 3.5. The quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ is the superalgebra defined by

(R1) generators $s_{ij}^{(n)}, t_{ij}^{(n)}$ for $i, j \in I$ and $n \in \mathbb{Z}_{\geq 0}$;

(R2) \mathbb{Z}_2 -grading $|s_{ij}^{(n)}| = |t_{ij}^{(n)}| = |i| + |j|$;

(R3) *RTT-relations* [FRT89]

$$(3.9) \quad R_{23}(z, w)T_{12}(z)T_{13}(w) = T_{13}(w)T_{12}(z)R_{23}(z, w),$$

$$(3.10) \quad R_{23}(z, w)S_{12}(z)S_{13}(w) = S_{13}(w)S_{12}(z)R_{23}(z, w),$$

$$(3.11) \quad R_{23}(z, w)T_{12}(z)S_{13}(w) = S_{13}(w)T_{12}(z)R_{23}(z, w),$$

$$(3.12) \quad t_{ij}^{(0)} = s_{ji}^{(0)} = 0 \quad \text{for } 1 \leq i < j \leq M + N,$$

$$(3.13) \quad t_{ii}^{(0)} s_{ii}^{(0)} = 1 = s_{ii}^{(0)} t_{ii}^{(0)} \quad \text{for } i \in I.$$

Here $T(z) = \sum_{i,j \in I} t_{ij}(z) \otimes E_{ij} \in (U_q(\widehat{\mathfrak{g}}) \otimes \text{End}\mathbf{V})[[z^{-1}]]$ and $t_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} t_{ij}^{(n)} z^{-n} \in U_q(\widehat{\mathfrak{g}})[[z^{-1}]]$ (similar convention for $S(z)$ with the z^{-n} replaced by the z^n).

The q -Yangian $Y_q(\mathfrak{g})$ is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by the $(s_{ii}^{(0)})^{-1}, s_{ij}^{(n)}$ for $i, j \in I$ and $n \in \mathbb{Z}_{\geq 0}$.

As in §3.1.1, $U_q(\widehat{\mathfrak{g}})$ is endowed with a super bialgebra structure:

$$(3.14) \quad \Delta(s_{ij}^{(n)}) = \sum_{a=0}^n \sum_{k \in I} (-1)^{(|i|+|k|)(|k|+|j|)} s_{ik}^{(a)} \otimes s_{kj}^{(n-a)},$$

$$(3.15) \quad \Delta(t_{ij}^{(n)}) = \sum_{a=0}^n \sum_{k \in I} (-1)^{(|i|+|k|)(|k|+|j|)} t_{ik}^{(a)} \otimes t_{kj}^{(n-a)}.$$

Indeed, the antipode $\mathbb{S} : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})$ exists:

$$(3.16) \quad \sum_{i,j \in I} \mathbb{S}(s_{ij}(z)) \otimes E_{ij} = \left(\sum_{i,j \in I} s_{ij}(z) \otimes E_{ij} \right)^{-1} \in (U_q(\widehat{\mathfrak{g}}) \otimes \text{End} \mathbf{V})[[z]],$$

$$(3.17) \quad \sum_{i,j \in I} \mathbb{S}(t_{ij}(z)) \otimes E_{ij} = \left(\sum_{i,j \in I} t_{ij}(z) \otimes E_{ij} \right)^{-1} \in (U_q(\widehat{\mathfrak{g}}) \otimes \text{End} \mathbf{V})[[z^{-1}]].$$

Here the RHS of the above formulas are well defined thanks to Relation (3.12). It follows that $U_q(\widehat{\mathfrak{g}})$ is a Hopf superalgebra, with $U_q(\mathfrak{g})$ a sub-Hopf-superalgebra.

3.3. Structure of the quantum affine superalgebra. We study the weight grading, the \mathbb{Z} -grading and evaluation morphisms for $U_q(\widehat{\mathfrak{g}})$. Moreover, we explain that $U_q(\widehat{\mathfrak{g}})$ is indeed a quantum double associated with a Hopf pairing.

3.3.1. The Perk-Schultz R -matrix. The exact form of $R(z, w) \in (\text{End} \mathbf{V} \otimes \text{End} \mathbf{V})[z, w]$ is:

$$(3.18) \quad \begin{aligned} R(z, w) = & \sum_{i \in I} (zq_i - wq_i^{-1}) E_{ii} \otimes E_{ii} + (z - w) \sum_{i \neq j} E_{ii} \otimes E_{jj} \\ & + z \sum_{i < j} (q_i - q_i^{-1}) E_{ji} \otimes E_{ij} + w \sum_{i < j} (q_j - q_j^{-1}) E_{ij} \otimes E_{ji}. \end{aligned}$$

Let us gather the following fundamental properties of the Perk-Schultz R -matrix $R(z, w)$.

Proposition 3.6. *The Perk-Schultz R -matrix $R(z, w)$ verifies the following relations.*

- (PS1) *Yang-Baxter:* $R_{12}(z_1, z_2) R_{13}(z_1, z_3) R_{23}(z_2, z_3) = R_{23}(z_2, z_3) R_{13}(z_1, z_3) R_{12}(z_1, z_2)$.
- (PS2) *Unitarity relation:* $R(z, w) c_{\mathbf{V}, \mathbf{V}} R(w, z) c_{\mathbf{V}, \mathbf{V}} = (zq - wq^{-1})(wq - zq^{-1}) \text{Id}_{\mathbf{V} \otimes \mathbf{V}}$.
- (PS3) *Ice rule:* $R_{ab, cd}(z, w) \neq 0 \implies \epsilon_a + \epsilon_b = \epsilon_c + \epsilon_d \in \mathbf{P}$ for $a, b, c, d \in I$.
- (PS4) $c_{\mathbf{V}, \mathbf{V}} R(z, w) c_{\mathbf{V}, \mathbf{V}} = (\tau \otimes \tau)(R(z, w)) \in (\text{End} \mathbf{V})^{\otimes 2}[z, w]$.
- (PS5) *Let $R = R(1, 0)$, $R' = c_{\mathbf{V}, \mathbf{V}} R^{-1} c_{\mathbf{V}, \mathbf{V}}$. Then $R(z, w) = zR - wR'$.*
- (PS6) *Hecke relation:* $R' = R - (q - q^{-1}) c_{\mathbf{V}, \mathbf{V}}$.

Here $R_{ab, cd}(z, w) \in \mathbb{C}[z, w]$ for $a, b, c, d \in I$ are the matrix elements defined by

$$R(z, w)(v_c \otimes v_d) = \sum_{a, b \in I} R_{ab, cd}(z, w)(v_a \otimes v_b) \in \mathbf{V}^{\otimes 2}[z, w].$$

Proof. (PS1)-(PS3) have been observed in [PS81]. (PS4) comes from Formula (3.18). (PS5) follows from (PS2) and Formula (3.8). For (PS6), note that

$$c_{\mathbf{V}, \mathbf{V}} R = qP^+ - q^{-1}P^-$$

gives a diagonal decomposition for the matrix $c_{\mathbf{V}, \mathbf{V}} R \in \text{End}(\mathbf{V}^{\otimes 2})$. Henceforth

$$(c_{\mathbf{V}, \mathbf{V}} R)^2 = (q - q^{-1}) c_{\mathbf{V}, \mathbf{V}} R + \text{Id}_{\mathbf{V} \otimes \mathbf{V}}, \quad (c_{\mathbf{V}, \mathbf{V}} R)^{-1} = c_{\mathbf{V}, \mathbf{V}} R - (q - q^{-1}) \text{Id}_{\mathbf{V} \otimes \mathbf{V}},$$

from which the Hecke relation follows. \square

Remark 3.7. (1) Later we shall care about the parameter q . In this case, write $R(z, w) = R_q(z, w)$, $R = R_q$, $R' = R'_q$. Define

$$\mathcal{R}(z, w) = \mathcal{R}_q(z, w) := \frac{R(z, w)}{zq - wq^{-1}}.$$

Then the inverses of these matrices $R_q, R'_q, R_q(z, w)$ have the following simple expression:

$$(PS7) \quad (R_q)^{-1} = R_{q^{-1}}, \quad (R'_q)^{-1} = R'_{q^{-1}}, \quad \mathcal{R}_q(z, w)^{-1} = \mathcal{R}_{q^{-1}}(z, w).$$

(2) Remark that in Definition 3.5 of $U_q(\widehat{\mathfrak{g}})$, one can replace $R(z, w)$ by $\mathcal{R}_q(z, w)$ everywhere. As for Relation (3.11), $\mathcal{R}_q(z, w)$ should be viewed as a formal power series in $\frac{w}{z}$.

(3) From Proposition 3.4 and (PS4) follows an isomorphism of Hopf superalgebras

$$(3.19) \quad \Psi : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\text{cop}}, \quad s_{ij}^{(n)} \mapsto \varepsilon_{ji} t_{ji}^{(n)}, \quad t_{ij}^{(n)} \mapsto \varepsilon_{ji} s_{ji}^{(n)}.$$

3.3.2. *\mathbb{Z} -grading.* There exists a \mathbb{Z} -grading on $U_q(\widehat{\mathfrak{g}})$:

$$(3.20) \quad |s_{ij}^{(n)}|_{\mathbb{Z}} = n, \quad |t_{ij}^{(n)}|_{\mathbb{Z}} = -n.$$

This \mathbb{Z} -grading is compatible with the Hopf superalgebra structure. In particular, we get an one-parameter family $(\Phi_a : a \in \mathbb{C}^\times)$ of Hopf superalgebra automorphisms:

$$(3.21) \quad \Phi_a : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad s_{ij}^{(n)} \mapsto a^n s_{ij}^{(n)}, \quad t_{ij}^{(n)} \mapsto a^{-n} t_{ij}^{(n)}.$$

The main reason behind this \mathbb{Z} -grading is that $\mathcal{R}(az, aw) = \mathcal{R}(z, w)$ for all $a \in \mathbb{C}^\times$.

3.3.3. *Automorphisms given by power series.* Let $f(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$ and $g(z) \in 1 + z\mathbb{C}[[z]]$. There exists an automorphism of superalgebra:

$$(3.22) \quad \phi_{(f(z), g(z))} : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad t_{ij}(z) \mapsto f(z)t_{ij}(z), \quad s_{ij}(z) \mapsto g(z)s_{ij}(z).$$

These automorphisms behave well under coproduct in the following way:

$$(\phi_{(f_1(z), g_1(z))} \otimes \phi_{(f_2(z), g_2(z))}) \circ \Delta = \Delta \circ \phi_{(f_1(z)f_2(z), g_1(z)g_2(z))} : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\otimes 2}$$

for $f_1(z), f_2(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$, $g_1(z), g_2(z) \in 1 + z\mathbb{C}[[z]]$.

Note that $\phi_{(f(z), g(z))}$ restricts to an automorphism of q -Yangian: $\phi_g : Y_q(\mathfrak{g}) \longrightarrow Y_q(\mathfrak{g})$.

3.3.4. *Evaluation morphisms.* Recall the R -matrix $R \in (\text{End } \mathbf{V})^{\otimes 2}$ in Proposition 3.6. As in Definition 3.5, let us define $\mathcal{U}_q(\mathfrak{g})$ to be the superalgebra generated by s_{ij}, t_{ji} for $1 \leq i \leq j \leq M + N$, with \mathbb{Z}_2 -degrees

$$|s_{ij}| = |t_{ji}| = |i| + |j|$$

and with RTT relations ([FRT90])

$$\begin{aligned} R_{23}T_{12}T_{13} &= T_{13}T_{12}R_{23}, & R_{23}S_{12}S_{13} &= S_{13}S_{12}R_{23} \\ R_{23}T_{12}S_{13} &= S_{13}T_{12}R_{23}, & s_{ii}t_{ii} &= 1 = t_{ii}s_{ii}. \end{aligned}$$

Here, as usual, $T = \sum_{i \leq j} t_{ji} \otimes E_{ji}$, $S = \sum_{i \leq j} s_{ij} \otimes E_{ij} \in \mathcal{U}_q(\mathfrak{g}) \otimes \text{End } \mathbf{V}$. $\mathcal{U}_q(\mathfrak{g})$ is endowed with a Hopf superalgebra structure with similar coproduct as in formulas (3.14)-(3.15).

Proposition 3.8. (1) The assignment $s_{ij} \mapsto s_{ij}^{(0)}$, $t_{ji} \mapsto t_{ji}^{(0)}$ extends uniquely to a Hopf superalgebra morphism $\iota : \mathcal{U}_q(\mathfrak{g}) \longrightarrow U_q(\widehat{\mathfrak{g}})$.

(2) The assignment $s_{ij}(z) \mapsto s_{ij} - zt_{ij}$, $t_{ij}(z) \mapsto t_{ij} - z^{-1}s_{ij}$ extends uniquely to a superalgebra morphism $\text{ev} : U_q(\widehat{\mathfrak{g}}) \longrightarrow \mathcal{U}_q(\mathfrak{g})$.

We understand that $s_{ji} = t_{ij} = 0$ in the superalgebra $\mathcal{U}_q(\mathfrak{g})$ for $1 \leq i < j \leq M + N$. The morphism ev is called an *evaluation morphism*. It is clear that $\text{ev} \circ \iota = \text{Id}_{\mathcal{U}_q(\mathfrak{g})}$.

Proof. (1) To show that ι is well defined, we only need to check that in $U_q(\widehat{\mathfrak{g}}) \otimes \text{End} \mathbf{V}$,

$$\begin{aligned} R_{23}T_{12}(\infty)T_{13}(\infty) &= T_{13}(\infty)T_{12}(\infty)R_{23}, & R_{23}S_{12}(0)T_{13}(\infty) &= T_{13}(\infty)S_{12}(0)R_{23}, \\ R_{23}S_{12}(0)S_{13}(0) &= S_{13}(0)S_{12}(0)R_{23}. \end{aligned}$$

By comparing the coefficients of z in both sides of Relations (3.9),(3.11),(3.10), we get the above three equations respectively.

(2) As before, we only need to check that ev respects Relations (3.9)-(3.11) (the other relations are obvious). We do this for Relation (3.11), the other two being analogous. In other words, we need to show that in the superalgebra $(\mathcal{U}_q(\mathfrak{g}) \otimes (\text{End} \mathbf{V})^{\otimes 2})[z, z^{-1}, w]$,

$$(zR_{23} - wR'_{23})(T_{12} - z^{-1}S_{12})(S_{13} - wT_{13}) = (S_{13} - wT_{13})(T_{12} - z^{-1}S_{12})(zR_{23} - wR'_{23}).$$

Note that Proposition 3.4 says that in $\mathcal{U}_q(\mathfrak{g}) \otimes (\text{End} \mathbf{V})^{\otimes 2}$

$$R'_{23}S_{12}S_{13} = S_{13}S_{12}R'_{23}, \quad R'_{23}T_{12}T_{13} = T_{13}T_{12}R'_{23}, \quad R'_{23}S_{12}T_{13} = T_{13}S_{12}R'_{23}.$$

By comparing the coefficients of both sides, we are left to verify the following:

$$R_{23}S_{12}T_{13} - T_{13}S_{12}R_{23} = R'_{23}T_{12}S_{13} - S_{13}T_{12}R'_{23}.$$

By using the Hecke relation (PS6) in Proposition 3.6 we get (writing $c = c_{\mathbf{V}, \mathbf{V}}$)

$$\begin{aligned} RHS &= (R_{23} - (q - q^{-1})c_{23})T_{12}S_{13} - S_{13}T_{12}(R_{23} - (q - q^{-1})c_{23}) \\ &= (R_{23}T_{12}S_{13} - S_{13}T_{12}R_{23}) - (q - q^{-1})c_{23}T_{12}S_{13} + (q - q^{-1})S_{13}T_{12}c_{23} \\ &= LHS + T_{13}S_{12}R_{23} - (q - q^{-1})c_{23}T_{12}S_{13} - R_{23}S_{12}T_{13} + (q - q^{-1})S_{13}T_{12}c_{23} \\ &= LHS + T_{13}S_{12}(R_{23} - (q - q^{-1})c_{23}) - (R_{23} - (q - q^{-1})c_{23})S_{12}T_{13} \\ &= LHS + T_{13}S_{12}R'_{23} - R'_{23}S_{12}T_{13} = LHS. \end{aligned}$$

In the above, we used that $c_{23}T_{1i} = T_{1j}c_{23}$ for $\{i, j\} = \{2, 3\}$. This concludes the proof. \square

In the above proof, the quadratic Hecke relation (PS6) has been used repeatedly in an essential way. This may gives an explanation of the fact: for \mathfrak{g}' a complex simple finite-dimensional Lie algebra, evaluation morphisms $\text{ev} : U_q(\widehat{\mathfrak{g}}') \rightarrow U_q(\mathfrak{g}')$ exist only in type A. (Even in type A, the image of ev is contained in certain enlargement of $U_q(\mathfrak{g}')$.)

3.3.5. Isomorphisms between $U_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$. As their notations suggest, the two Hopf superalgebras $U_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$ should be isomorphic.

Proposition 3.9. *There is an isomorphism of Hopf superalgebras $DF : U_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$*

$$e_i^+ \mapsto \frac{s_{ii}^{-1}s_{i,i+1}}{1 - q_i^{-2}}, \quad e_i^- \mapsto \frac{t_{i+1,i}t_{ii}^{-1}}{1 - q_i^2}, \quad t_j^{d,j} \mapsto s_{jj} = t_{jj}^{-1} \quad (i \in I_0, j \in I).$$

We postpone the proof of this proposition to §3.4.2.

In the above formulas, the scalars are chosen in such a way that the natural representation of $\mathcal{U}_q(\mathfrak{g})$ on \mathbf{V} defined later will match perfectly with that of $U_q(\mathfrak{g})$ (see Example 1 and §4.4). We list the following relations in the superalgebra $\mathcal{U}_q(\mathfrak{g})$ to be used later:

$$\begin{aligned} [s_{i,i+1}, t_{j+1,j}] &= \delta_{ij}(q_i - q_i^{-1})(t_{ii}s_{i+1,i+1} - s_{ii}t_{i+1,i+1}) \quad i, j \in I_0, \\ [t_{ji}, t_{kj}] &= (q_j - q_j^{-1})t_{jj}t_{ki} \quad i, j, k \in I, i < j < k. \end{aligned}$$

The second equation above is deduced in the proof of Lemma A.1. See Equation (A.29). The first equation will follow from Theorem 3.12.

3.3.6. Quantum double construction. We reformulate the RTT definition of $U_q(\widehat{\mathfrak{g}})$ as a quantum double construction, as in the non-graded case [FRT90, Theorem 16]. This will in turn give a RTT presentation of the q -Yangian $Y_q(\mathfrak{g})$ in Definition 3.5.

Let A, B be two Hopf superalgebras. Call a bilinear form $\varphi : A \times B \longrightarrow \mathbb{C}$ a *Hopf pairing* if φ is of \mathbb{Z}_2 -degree $\overline{0}$, and if φ satisfies

$$\begin{aligned}\varphi(a, bb') &= (-1)^{|b||b'|} \varphi(a_{(1)}, b) \varphi(a_{(2)}, b'), & \varphi(a, 1) &= \varepsilon_A(a); \\ \varphi(aa', b) &= \varphi(a', b_{(1)}) \varphi(a, b_{(2)}), & \varphi(1, b) &= \varepsilon_B(b)\end{aligned}$$

for \mathbb{Z}_2 -homogeneous $a, a' \in A$ and $b, b' \in B$. Here we adapt the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$. Given such a Hopf pairing, one can endow the vector superspace $A \otimes B$ with a unique Hopf superalgebra structure satisfying [KRT97, Theorem 3.2]

(QD1) $a \mapsto a \otimes 1$, $b \mapsto 1 \otimes b$ are morphisms of Hopf superalgebras respectively;

(QD2) for \mathbb{Z}_2 -homogeneous $a \in A, b \in B$, we have $(a \otimes 1)(1 \otimes b) = a \otimes b$ and

$$(1 \otimes b)(a \otimes 1) = (-1)^{|a_{(1)}||b| + (|b_{(2)}| + |b_{(3)}|)|a_{(2)}| + |a_{(3)}||b_{(3)}|} \varphi(a_{(1)}, S_B(b_{(1)})) \varphi(a_{(3)}, b_{(3)}) a_{(2)} \otimes b_{(2)}.$$

Let $\mathcal{D}_\varphi(A, B)$ be the Hopf superalgebra thus obtained.

In our context, A (resp. B) is the superalgebra generated by the $s_{ij}^{(n)}, (s_{ii}^{(0)})^{-1}$ (resp. the $t_{ij}^{(n)}, (t_{ii}^{(0)})^{-1}$) with \mathbb{Z}_2 -gradings and with defining relations as in Definition 3.5 (without Relation (3.11) which makes no sense). Clearly A and B are Hopf superalgebras with coproducts defined by formulas (3.14)-(3.15).

Proposition 3.10. *There exists uniquely a Hopf pairing $\widehat{\varphi} : A \times B \longrightarrow \mathbb{C}$ such that*

$$(3.23) \quad \sum_{i,j,a,b \in I} E_{ab} \otimes E_{ij} \sum_{m,n \in \mathbb{Z}_{\geq 0}} z^{-m} w^n \widehat{\varphi}(s_{ij}^{(n)}, t_{ab}^{(m)}) = \mathcal{R}(z, w) \in (\text{End } V)^{\otimes 2}[[z^{-1}, w]].$$

The assignment $s_{ij}^{(n)} \otimes 1 \mapsto s_{ij}^{(n)}$, $1 \otimes t_{ij}^{(n)} \mapsto t_{ij}^{(n)}$ extends uniquely to a surjective morphism of Hopf superalgebras $D : \mathcal{D}_{\widehat{\varphi}}(A, B) \longrightarrow U_q(\widehat{\mathfrak{g}})$ whose kernel is the ideal generated by the

$$s_{ii}^{(0)} \otimes 1 - 1 \otimes (t_{ii}^{(0)})^{-1}, \quad 1 \otimes t_{ii}^{(0)} - (s_{ii}^{(0)})^{-1} \otimes 1 \quad (i \in I).$$

Moreover, D restricts to a Hopf superalgebra isomorphism $D|_A : A \longrightarrow Y_q(\mathfrak{g})$.

Proof. (Sketch) By abuse of language, let \mathcal{F}_A (resp. \mathcal{F}_B) be the superalgebra freely generated by the $s_{ij}^{(n)}$ (resp. the $t_{ij}^{(n)}$) for $i, j \in I, n \in \mathbb{Z}_{\geq 0}$, and with \mathbb{Z}_2 -gradings $|s_{ij}^{(n)}| = |t_{ij}^{(n)}| = |i| + |j|$. Then \mathcal{F}_A and \mathcal{F}_B are super bialgebras with coproduct given by Equations (3.14)-(3.15). Now Formula (3.23) above determines a bilinear form $\varphi : \mathcal{F}_A \times \mathcal{F}_B \longrightarrow \mathbb{C}$ satisfying all the properties of a Hopf pairing. According to [KRT97, Chapter 3] it is enough to show that φ respects Relations (3.9)-(3.10), (3.12)-(3.13), and that (QD2) is equivalent to Relation (3.11). We only check Relation (3.10). (The other relations can be done in the same way.) For this, define the bilinear map

$$\begin{cases} \varphi_3 : \mathcal{F}_A^{\otimes 2} \otimes (\text{End } V)^{\otimes 2} \times \mathcal{F}_B^{\otimes 2} \otimes \text{End } V \longrightarrow (\text{End } V)^{\otimes 3} \\ (a \otimes a' \otimes x \otimes y, b \otimes b' \otimes z) \mapsto (-1)^{(|x|+|y|)(|b|+|b'|+|z|)+|a'||b|} \varphi(a, b) \varphi(a', b') z \otimes x \otimes y. \end{cases}$$

for \mathbb{Z}_2 -homogeneous vectors a, a', x, y, z, b, b' . Then Relation (3.10) amounts to:

$$\varphi_3(R_{34}(z, w)S_{13}(z)S_{24}(w) - S_{14}(w)S_{23}(z)R_{34}(z, w), T_{23}(u)T_{13}(u)) = 0.$$

From the definitions of φ and φ_3 , we see that the LHS of the above equation becomes:

$$R_{23}(z, w)\mathcal{R}_{13}(u, w)\mathcal{R}_{12}(u, z) - \mathcal{R}_{12}(u, z)\mathcal{R}_{13}(u, w)R_{23}(z, w),$$

which is zero because of the Yang-Baxter Equation 3.6 (PS1). \square

3.3.7. Weight grading. The following relations hold in $U_q(\widehat{\mathfrak{g}})$:

$$(3.24) \quad s_{ii}^{(0)} s_{jk}^{(n)} = q^{(\epsilon_i, \epsilon_j - \epsilon_k)} s_{jk}^{(n)} s_{ii}^{(0)}, \quad s_{ii}^{(0)} t_{jk}^{(n)} = q^{(\epsilon_i, \epsilon_j - \epsilon_k)} t_{jk}^{(n)} s_{ii}^{(0)}.$$

As the $s_{ii}^{(0)}$ are invertible, $U_q(\widehat{\mathfrak{g}})$ is endowed with a \mathbf{Q} -grading: for $\alpha \in \mathbf{Q}$,

$$(U_q(\widehat{\mathfrak{g}}))_\alpha = \{x \in U_q(\widehat{\mathfrak{g}}) \mid s_{ii}^{(0)} x (s_{ii}^{(0)})^{-1} = q^{(\epsilon_i, \alpha)} x \text{ for } i \in I\}.$$

The \mathbf{Q} -grading is compatible with the Hopf superalgebra structure and with the \mathbb{Z}_2 -grading:

$$(3.25) \quad |s_{ij}^{(n)}|_{\mathbf{Q}} = |t_{ij}^{(n)}|_{\mathbf{Q}} = \epsilon_i - \epsilon_j \quad (i, j \in I).$$

Let us prove Equation (3.24). To begin with, for $n \in \mathbb{Z}_{\geq 0}$, by taking the coefficients of z^{n+1} (resp. z^{1-n}) in Relation (3.10) (resp. Relation (3.11)), we observe that

$$R_{23}S_{12}^{(n)}S_{13}^{(0)} = S_{13}^{(0)}S_{12}^{(n)}R_{23}, \quad R_{23}T_{12}^{(n)}S_{13}^{(0)} = S_{13}^{(0)}T_{12}^{(n)}R_{23}.$$

Here $S^{(n)} := \sum_{ij} s_{ij}^{(n)} \otimes E_{ij} \in U_q(\widehat{\mathfrak{g}}) \otimes E_{ij}$ (similar for $T^{(n)}$). Now Equation (3.24) comes from the following lemma and from the automorphism defined by Equation (3.19).

Lemma 3.11. *Let U be a superalgebra. For $i, j \in I$ let $a_{ij}, b_{ij} \in U$ be elements of \mathbb{Z}_2 -degree $|i| + |j|$. Assume that $b_{ij} = 0$ if $i > j$. Introduce*

$$A := \sum_{i,j \in I} a_{ij} \otimes E_{ij}, \quad B := \sum_{i,j \in I} b_{ij} \otimes E_{ij} \in U \otimes \text{End } \mathbf{V}.$$

Suppose that $R_{23}A_{12}B_{13} = B_{13}A_{12}R_{23}$. Then

$$b_{kk}a_{ij} = q^{(\epsilon_k, \epsilon_i - \epsilon_j)} a_{ij}b_{kk} \quad \text{for } i, j, k \in I.$$

Proof. We shall prove only the case $k \neq i, j$. The idea is to compare the matrix coefficients of $v_j \otimes v_k \mapsto v_i \otimes v_k$ for the operator equation

$$A_{12}B_{13} = R_{23}^{-1}B_{13}A_{12}R_{23} \in U \otimes \text{End } \mathbf{V}^{\otimes 2}.$$

For a statement P define $\delta(P) = 1$ if P is true and 0 otherwise. The matrix coefficient for the LHS is $a_{ij}b_{kk}$. For the RHS, in view of the ice rule for R , we see that the corresponding coefficient c should be the coefficient c_1 of $v_i \otimes v_k$ in

$$R_{23}^{-1}B_{13}A_{12}(v_j \otimes v_k + \delta(j < k)(q_j - q_j^{-1})v_k \otimes v_j).$$

To determine c_1 , it is enough to consider the part u_1 containing $v_i \otimes v_k, v_k \otimes v_i$ in the vector

$$B_{13}A_{12}(v_j \otimes v_k + \delta(j < k)(q_j - q_j^{-1})v_k \otimes v_j)$$

A straightforward calculation shows that

$$u_1 = (b_{kk}a_{ij} \pm \delta(j < k)(q_j - q_j^{-1})b_{kj}a_{ik})v_i \otimes v_k + (\delta(j < k)(q_j - q_j^{-1})b_{ij}a_{kk} \pm b_{ik}a_{kj})v_k \otimes v_i.$$

Observe that the last three terms in u_1 do not contribute to $v_i \otimes v_k$ when applying R_{23}^{-1} . It follows that the coefficient $c = c_1$ of $v_i \otimes v_k$ in $R_{23}^{-1}u_1$ is exactly $b_{kk}a_{ij}$. In other words, $a_{ij}b_{kk} = b_{kk}a_{ij}$, as desired. \square

Remark that in a similar way, we can introduce the weight grading for $\mathcal{U}_q(\mathfrak{g})$:

$$|s_{ij}|_{\mathbf{Q}} = |t_{ji}|_{\mathbf{Q}} = \epsilon_i - \epsilon_j \quad \text{for } 1 \leq i \leq j \leq M + N.$$

The superalgebra morphisms ι, ev in Proposition 3.8 respect \mathbf{Q} -gradings.

3.4. Drinfeld realization and coproduct formulas. We explain that as Hopf superalgebras the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ is not far from the quantum loop superalgebra $U_q(L\mathfrak{g})$ defined by Drinfeld generators (such quantum loop superalgebra with \mathfrak{g} replaced by $\mathfrak{sl}(M, N)$ has been used in [Zh13] to study finite-dimensional representations). Also some coproduct estimations for the Drinfeld generators are given.

3.4.1. Ding-Frenkel homomorphism. We review a super analogue of Ding-Frenkel homomorphism between Drinfeld and RTT realizations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$ [DF93, pp.286], following Yao-Zhong Zhang [Zy97].

The Gauss decomposition gives uniquely

$$e_{ij}^{\pm}(z), f_{ji}^{\pm}(z), K_l^{\pm}(z) \in U_q(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \quad \text{for } 1 \leq i < j \leq M + N, 1 \leq l \leq M + N$$

such that in the superalgebra $(U_q(\widehat{\mathfrak{g}}) \otimes \text{End} \mathbf{V})[[z, z^{-1}]]$

$$\begin{cases} S(z) = (\sum_{i < j} f_{ji}^+(z) \otimes E_{ji} + 1 \otimes \text{Id}_{\mathbf{V}}) (\sum_l K_l^+(z) \otimes E_{ii}) (\sum_{i < j} e_{ij}^+(z) \otimes E_{ij} + 1 \otimes \text{Id}_{\mathbf{V}}), \\ T(z) = (\sum_{i < j} f_{ji}^-(z) \otimes E_{ji} + 1 \otimes \text{Id}_{\mathbf{V}}) (\sum_l K_l^-(z) \otimes E_{ii}) (\sum_{i < j} e_{ij}^-(z) \otimes E_{ij} + 1 \otimes \text{Id}_{\mathbf{V}}). \end{cases}$$

For example, $K_1^+(z) = s_{11}(z)$ and $K_1^-(z) = t_{11}(z)$. For $i \in I_0 = I \setminus \{M + N\}$, define

$$X_i^+(z) = e_{i,i+1}^+(z) - e_{i,i+1}^-(z) = \sum_n X_{i,n}^+ z^n, \quad X_i^-(z) = f_{i+1,i}^-(z) - f_{i+1,i}^+(z) = \sum_n X_{i,n}^- z^n.$$

Theorem 3.12. [DF93, Zy97] *The superalgebra $U_q(\widehat{\mathfrak{g}})$ is generated by the coefficients of $X_i^{\pm}(z), K_j^{\pm}(z)$ with $i \in I_0, j \in I$. Moreover,*

$$\begin{aligned} K_i^{\epsilon}(z) K_j^{\epsilon'}(w) &= K_j^{\epsilon'}(w) K_i^{\epsilon}(z), \\ (\text{Cartan}) \quad \begin{cases} K_i^{\epsilon}(z) X_j^{\epsilon'}(w) = X_j^{\epsilon'}(w) k_i^{\epsilon}(z) & \text{for } i \notin \{j, j+1\}, \\ K_i^{\epsilon}(z) X_i^{\pm}(w) = (\frac{q_i z - q_i^{-1} w}{z-w})^{\mp 1} X_i^{\pm}(w) K_i^{\epsilon}(z), \\ K_{i+1}^{\epsilon}(z) X_i^{\pm}(w) = (\frac{q_{i+1}^{-1} z - q_{i+1} w}{z-w})^{\mp 1} X_i^{\pm}(w) K_{i+1}^{\epsilon}(z), \end{cases} \\ (\text{Drinfeld}) \quad \begin{cases} X_i^{\epsilon}(z) X_j^{\epsilon}(w) - X_j^{\epsilon}(w) X_i^{\epsilon}(z) = 0 & \text{if } |i-j| \geq 2, \\ X_M^{\epsilon}(z) X_M^{\epsilon}(w) + X_M^{\epsilon}(w) X_M^{\epsilon}(z) = 0, \\ (q_i^{\mp 1} z - q_i^{\pm 1} w) X_i^{\pm}(z) X_i^{\pm}(w) = (q_i^{\pm 1} z - q_i^{\mp 1} w) X_i^{\pm}(w) X_i^{\pm}(z) & \text{if } i \neq M, \\ (q_{i+1} z - q_{i+1}^{-1} w) X_i^+(z) X_{i+1}^+(w) = (z-w) X_{i+1}^+(w) X_i^+(z), \\ (z-w) X_i^-(z) X_{i+1}^-(w) = (q_{i+1} z - q_{i+1}^{-1} w) X_{i+1}^-(w) X_i^-(z), \end{cases} \\ [X_i^+(z), X_j^-(w)] &= \delta_{ij} (q_i - q_i^{-1}) \delta(\frac{z}{w}) (K_{i+1}^+(z) K_i^+(z)^{-1} - K_{i+1}^-(w) K_i^-(w)^{-1}), \end{aligned}$$

$$(Serre) \begin{cases} [X_i^\epsilon(z_1), [X_i^\epsilon(z_2), X_j^\epsilon(w)]_q]_{q^{-1}} + \{z_1 \leftrightarrow z_2\} = 0 & \text{if } (i \neq M, |j - i| = 1), \\ [[X_{M-1}^\epsilon(u), X_M^\epsilon(z_1)]_q, X_{M+1}^\epsilon(v)]_{q^{-1}}, X_M^\epsilon(z_2)] + \{z_1 \leftrightarrow z_2\} = 0 & \text{if } (M, N > 1). \end{cases}$$

□

3.4.2. *Proof of Proposition 3.9.* As an application of Theorem 3.12 above, let us give a proof of Proposition 3.9 which says that the Ding-Frenkel homomorphism restricted to finite type quantum superalgebras is indeed an isomorphism.

First of all, $DF : U_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$ is a well-defined Hopf superalgebra homomorphism thanks to Theorem 3.12, Proposition 3.8 and [Ya94, Theorem 10.5.1] on defining relations of the quantum superalgebra $U_q(\mathfrak{g})$ of type A. It is easy to prove that DF is surjective in view of the relations preceding §3.3.6.

Next, let A (resp. B) be the subalgebra of $U_q(\mathfrak{g})$ generated by the $e_i^+, t_j^{\pm 1}$ (resp. the e_i^- and $t_j^{\pm 1}$) for $i \in I_0, j \in I$. Then A, B are sub-Hopf-superalgebras. Moreover, according to [Ya94, §2.4, Prop.10.4.1], there exists a non-degenerate Hopf pairing $\varphi_1 : A \times B \rightarrow \mathbb{C}$ defined by:

$$\varphi_1(t_i, t_j) = q^{-(\epsilon_i, \epsilon_j)}, \quad \varphi_1(e_i^+, e_j^-) = \frac{\delta_{ij}}{q_i^{-1} - q_i}.$$

Furthermore, $U_q(\mathfrak{g})$ is the quotient of $\mathcal{D}_{\varphi_1}(A, B)$ by the ideal generated by $1 \otimes t_i^{\pm 1} - t_i^{\pm 1} \otimes 1$.

On the other hand, let A' (resp. B') the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the s_{ij}, s_{kk}^{-1} (resp. the t_{ji}, t_{kk}^{-1}) for $1 \leq i \leq j \leq M + N$ and $k \in I$. Clearly A' and B' are sub-Hopf-superalgebras. Moreover Propositions 3.8 and 3.10 say that there exists a Hopf pairing $\varphi_2 : A' \times B' \rightarrow \mathbb{C}$ given by

$$\sum_{a,b,i,j \in I} \varphi_2(s_{ij}, t_{ab}) E_{ab} \otimes E_{ij} = R \in \text{End } \mathbf{V}^{\otimes 2}.$$

Similarly, $\mathcal{U}_q(\mathfrak{g})$ is the quotient of $\mathcal{D}_{\varphi_2}(A', B')$ by the ideal generated by $1 \otimes s_{ii}^{\pm 1} - s_{ii}^{\pm 1} \otimes 1$.

It is straightforward to show that $DF(A) = A'$ and $DF(B) = B'$. Moreover,

$$\varphi_2(DF(a), DF(b)) = \varphi_1(a, b) \quad \text{for } a \in A, b \in B.$$

Let $f : A \rightarrow A'$ (resp. $g : B \rightarrow B'$) be the Hopf superalgebra morphism induced by DF . Then f, g are surjective. Moreover, DF is induced by the Hopf superalgebra morphism

$$\mathcal{DF} := f \otimes g : \mathcal{D}_{\varphi_1}(A, B) \rightarrow \mathcal{D}_{\varphi_2}(A', B'), \quad a \otimes b \mapsto f(a) \otimes g(b).$$

As $DF(t_i) = s_{ii}^{d_i}$ for $i \in I$, we are reduced to show that \mathcal{DF} is injective. Note that

$$\ker \mathcal{DF} = \ker(f \otimes g) = \ker f \otimes B + A \otimes \ker g.$$

The non-degeneracy of φ_1 implies that the RHS above is zero. □

We remark that φ_2 defined above is non-degenerate. Hence we can write down the universal R -matrix of $\mathcal{U}_q(\mathfrak{g})$ [Ya94, Theorem 10.6.1] in terms of the RTT generators. Similar arguments should apply to the affine case, which however requires additional information on some central elements of $U_q(\widehat{\mathfrak{g}})$, the so-called *quantum Berezinians*, and their behaviour under the Hopf pairing $\widehat{\varphi}$. We hope to return to these issues in future works.

3.4.3. *Coproduct formulas.* Let us define the Drinfeld generators $K_{i,\pm s}^\pm$ for $s \in \mathbb{Z}_{\geq 0}$ by

$$K_i^\pm(z) = \sum_{s \in \mathbb{Z}_{\geq 0}} K_{i,\pm s}^\pm z^{\pm s} \in U_q(\widehat{\mathfrak{g}})[[z^{\pm 1}]].$$

Then $K_{i,0}^\pm = (s_{ii}^{(0)})^{\pm 1}$. Moreover Cartan relations in Theorem 3.12 imply that

$$|K_{i,n}^\pm|_{\mathbf{Q}} = \pm \alpha_i, \quad |K_{j,\pm s}^\pm|_{\mathbf{Q}} = 0 \quad \text{for } i \in I_0, j \in I, n \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}.$$

Proposition 3.13. *Let $i \in I_0, j \in I, n \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}$. Then*

$$(3.26) \quad \Delta(K_{j,\pm s}^\pm) - \sum_{a=0}^s K_{j,\pm a}^\pm \otimes K_{j,\pm(s-a)}^\pm \in \sum_{\alpha \in \mathbf{Q}_{\geq 0} \setminus \{0\}} (U_q(\widehat{\mathfrak{g}}))_\alpha \otimes (U_q(\widehat{\mathfrak{g}}))_{-\alpha},$$

$$(3.27) \quad \Delta(X_{i,n}^+) - 1 \otimes X_{i,n}^+ \in \sum_{\alpha \in \mathbf{Q}_{\geq 0} \setminus \{0\}} (U_q(\widehat{\mathfrak{g}}))_\alpha \otimes (U_q(\widehat{\mathfrak{g}}))_{\alpha_i - \alpha},$$

$$(3.28) \quad \Delta(X_{i,n}^-) - X_{i,n}^- \otimes 1 \in \sum_{\alpha \in \mathbf{Q}_{\geq 0} \setminus \{0\}} (U_q(\widehat{\mathfrak{g}}))_{\alpha - \alpha_i} \otimes (U_q(\widehat{\mathfrak{g}}))_{-\alpha}.$$

The proof of this proposition is given in Appendix A.

4. HIGHEST WEIGHT REPRESENTATIONS

In this section, we state one of the main results in this paper: some tensor products of Kirillov-Reshetikhin modules over $U_q(\widehat{\mathfrak{g}})$ are highest ℓ -weight modules.

4.1. **Highest ℓ -weight modules.** Let V be a $U_q(\widehat{\mathfrak{g}})$ -module. A non-zero vector $v \in V \setminus \{0\}$ is called a *highest ℓ -weight vector* if v is \mathbb{Z}_2 -homogeneous and

$$s_{ij}^{(n)} v = 0 = t_{ij}^{(n)} v, \quad s_{kk}^{(n)} v, t_{kk}^{(n)} v \in \mathbb{C}v \quad (n \in \mathbb{Z}_{\geq 0}, i, j, k \in I, i < j).$$

V is called a *highest ℓ -weight module* if $V = U_q(\widehat{\mathfrak{g}})v$ for some highest ℓ -weight vector. Similarly, the notions of *lowest ℓ -weight vector* and *lowest ℓ -weight module* are defined by replacing the above condition ($i < j$) with ($i > j$). Similarly, one can define the notions of highest/lowest ℓ -weight modules/vectors for representations of the q -Yangian $Y_q(\mathfrak{g})$ by dropping the $t_{ij}^{(n)}$ above.

Thanks to Proposition 3.9, there is a highest weight representation theory for the quantum superalgebra $\mathcal{U}_q(\mathfrak{g})$. For example, let V be a $\mathcal{U}_q(\mathfrak{g})$ -module. A non-zero vector $v \in V$ is called a *highest weight vector* if v is \mathbb{Z}_2 -homogeneous and $s_{ij}v = 0, s_{kk}v \in \mathbb{C}v$ for $i, j, k \in I, i < j$. In particular, for $\lambda \in \mathbf{P}$, we have simple $\mathcal{U}_q(\mathfrak{g})$ -module $(DF^{-1})^*L(\lambda)$ which will be written as $L(\lambda)$ by abuse of language. More explicitly, $L(\lambda)$ is the simple $\mathcal{U}_q(\mathfrak{g})$ -module generated by a vector v_λ such that

$$|v_\lambda| = |\lambda|, \quad s_{ij}v_\lambda = 0, \quad s_{kk}v_\lambda = q^{(\epsilon_k, \lambda)} v_\lambda \quad (i, j, k \in I, i < j).$$

4.1.1. *Highest ℓ -weights and tensor product.* Let V, V' be $U_q(\widehat{\mathfrak{g}})$ -modules of highest ℓ -weights with v, v' highest ℓ -weight vectors respectively. Then $v \otimes v'$ is also a highest ℓ -weight vector. By definition, there exist $f_i^\pm(z), g_i^\pm(z) \in (\mathbb{C}[[z^{\pm 1}]])^\times$ for $i \in I$ such that

$$s_{ii}(z)v = f_i^+(z)v, \quad t_{ii}(z)v = f_i^-(z)v, \quad s_{ii}(z)v' = g_i^+(z)v', \quad t_{ii}(z)v' = g_i^-(z)v'.$$

From the Gauss decomposition in §3.4.1, we see that

$$K_i^\pm(z)v = f_i^\pm(z)v, \quad K_i^\pm(z)v' = g_i^\pm(z)v'.$$

On the other hand, from the coproduct formulas of $s_{ii}(z), t_{ii}(z)$ it follows

$$s_{ii}(z)(v \otimes v') = f_i^+(z)g_i^+(z)(v \otimes v'), \quad t_{ii}(z)(v \otimes v') = f_i^-(z)g_i^-(z)(v \otimes v').$$

Henceforth, similar formulas hold for $K_i^\pm(z)(v \otimes v')$. This observation will be used in §A.3 to conclude the proof of Proposition 3.13.

4.1.2. *Kirillov-Reshetikhin modules.* For $a \in \mathbb{C}^\times$, define the evaluation morphism $\text{ev}_a := \text{ev} \circ \Phi_a : U_q(\widehat{\mathfrak{g}}) \rightarrow \mathcal{U}_q(\mathfrak{g})$, here ev and Φ_a are given by Proposition 3.8 and by Formula (3.21) respectively. We can pull back $\mathcal{U}_q(\mathfrak{g})$ -modules V to get $U_q(\widehat{\mathfrak{g}})$ -modules ev_a^*V . When there is no confusion, we simply write $v = \text{ev}_a^*v$ for $v \in V$.

For example, take $\lambda \in \mathbf{P}$. Consider $\text{ev}_a^*L(\lambda)$. Let v_λ be a highest weight vector for the $\mathcal{U}_q(\mathfrak{g})$ -module $L(\lambda)$, then v_λ is a highest ℓ -weight vector

$$|v_\lambda| = |\lambda|, \quad s_{ii}(z)v_\lambda = (q^{(\lambda, \epsilon_i)} - zaq^{-(\lambda, \epsilon_i)})v_\lambda, \quad t_{ii}(z)v_\lambda = (q^{-(\lambda, \epsilon_i)} - z^{-1}a^{-1}q^{(\lambda, \epsilon_i)})v_\lambda.$$

Definition 4.1. The $U_q(\widehat{\mathfrak{g}})$ -modules $\text{ev}_a^*L(k\varpi_r) \otimes \mathbb{C}_{|k\varpi_r|}$ for $a \in \mathbb{C}^\times, r \in I_0, k \in \mathbb{Z}_{\geq 0}$, are called Kirillov-Reshetikhin modules, denoted by $W_{k,a}^{(r)}$.

In the above definition, the tensor product by an one-dimensional module is needed to ensure that the highest ℓ -weight vectors are of \mathbb{Z}_2 -degree $\bar{0}$. The main result of this section is the following.

Theorem 4.2. Let $k \in \mathbb{Z}_{>0}$, $r \in I_0$ and $a \in \mathbb{C}^\times$. Let $x_j \in \mathbb{Z}$ for $1 \leq j \leq k$. Assume $x_i \geq x_j$ for all $1 \leq i < j \leq k$. Then the $U_q(\widehat{\mathfrak{g}})$ -module $\bigotimes_{j=1}^k W_{1, aq^{x_j}}^{(r)}$ is of highest ℓ -weight.

Large part of the rest of the paper is devoted to the proof of the theorem. The outline is as follows: first we reduce to the case $1 \leq r \leq M$ with the help of a Hopf superalgebra isomorphism (Remark 4.4)

$$U_q(\widehat{\mathfrak{gl}(N, M)})^{\text{cop}} \longrightarrow U_q(\widehat{\mathfrak{gl}(M, N)});$$

next we study in §5 in detail the case $\mathfrak{g} = \mathfrak{gl}(1, 1)$; finally we conclude in §6 the proof by restriction arguments from \mathfrak{g} to $\mathfrak{gl}(1, 1)$. Throughout the proof, a cyclicity result of Chari (Lemma 4.5) is used repeatedly.

4.2. Reduction to the case $r \leq M$. Let $\mathfrak{g}' = \mathfrak{gl}(N, M)$. We shall define the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}}')$. Recall that $U_q(\widehat{\mathfrak{g}})$ is constructed from the vector superspace \mathbf{V} and the Perk-Schultz matrix $R_q(z, w)$. We begin with the index set I endowed with \mathbb{Z}_2 -partition

$$I = \{1 < 2 < \cdots < M + N\} = I_{\overline{0}} \sqcup I_{\overline{1}}, \quad I_{\overline{0}} = \{1 < 2 < \cdots < M\}.$$

$\mathbf{V} = \bigoplus_{i \in I} \mathbb{C}v_i$ and $I = I_{\overline{0}} \sqcup I_{\overline{1}}$ are linked in the following way: $|v_i| = |i|$. The Perk-Schultz matrix $R(z, w)$ is determined by I as seen from Formula (3.18), in which the summation is over I , and $q_i = q^{(-1)^{|i|}}$.

Now introduce

$$J = \{1 < 2 < \cdots < M + N\} = J_{\overline{0}} \sqcup J_{\overline{1}}, \quad J_{\overline{0}} = \{1 < 2 < \cdots < N\}.$$

For $s \in \mathbb{Z}_2$ and $j \in J_s$, write $|j|^J = s$. Let $\mathbf{V}^J = \bigoplus_{j \in J} \mathbb{C}w_j$ be the vector superspace

$$|w_j| = |j|^J \quad (j \in J).$$

Let $e_{ij} \in \text{End} \mathbf{V}^J$ be $w_k \mapsto \delta_{jk} w_i$ for $i, j, k \in J$. Let $R^J(z, w) = R_q^J(z, w)$ be the Perk-Schultz matrix defined by Formula (3.18) with summation over J , with the q_i for $i \in I$ replaced by the $q_j^J = q^{(-1)^{|j|^J}}$ for $j \in J$, and with the E_{ij} for $i, j \in I$ replaced by the e_{ij} for $i, j \in J$.

Define the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}}')$ in exactly the same way as in Definition 3.5. For distinction, write the defining generators as $s_{ij;J}^{(n)}$, $t_{ij;J}^{(n)}$.

Finally, define ε_{ij}^J for $i, j \in J$ in the same way as in Formula (3.7), with the $|i|$ for $i \in I$ replaced by the $|i|^J$ for $i \in J$. For $i \in J$, let $\bar{i} = N + M + 1 - i \in I$.

Proposition 4.3. *The assignment $s_{ij;J}^{(n)} \mapsto \varepsilon_{ji}^J s_{ji}^{(n)}$, $t_{ij;J}^{(n)} \mapsto \varepsilon_{ji}^J t_{ji}^{(n)}$ extends uniquely to a Hopf superalgebra isomorphism $f_{J,I} : U_q(\widehat{\mathfrak{g}}')^{\text{cop}} \longrightarrow U_q(\widehat{\mathfrak{g}})$.*

Proof. Introduce the linear isomorphism $f : \mathbf{V} \longrightarrow \mathbf{V}^J$, $v_i \mapsto w_{\bar{i}}$. Let $f_* : \text{End} \mathbf{V} \longrightarrow \text{End} \mathbf{V}^J$, $h \mapsto fhf^{-1}$ be the induced map. Then

$$f_* : \text{End} \mathbf{V} \longrightarrow \text{End} \mathbf{V}^J, \quad E_{ij} \mapsto e_{\bar{i}\bar{j}}$$

is an isomorphism of superalgebras. Moreover, we have

$$f_* \otimes f_*(R_q(z, w)) = c_{\mathbf{V}^J, \mathbf{V}^J} R_{q^{-1}}^J(z, w) c_{\mathbf{V}^J, \mathbf{V}^J} = (\tau_J \otimes \tau_J)(R_q^J(z, w)^{-1}).$$

Here $\tau_J : \text{End} \mathbf{V}^J \longrightarrow \text{End} \mathbf{V}^J$, $e_{ij} \mapsto \varepsilon_{ij}^J e_{ji}$, and the last equation comes from Proposition 3.6 (PS4)-(PS5). Applying $\text{Id}_{U_q(\widehat{\mathfrak{g}})} \otimes f_* \otimes f_*$ to Relation (3.9), we get

$$((\tau_J \otimes \tau_J)(R_q^J(z, w)^{-1}))_{23} \tilde{T}_{12}(z) \tilde{T}_{13}(w) = \tilde{T}_{13}(w) \tilde{T}_{12}(z) ((\tau_J \otimes \tau_J)(R_q^J(z, w)^{-1}))_{23}.$$

Here $\tilde{T}(z) = (\text{Id}_{U_q(\widehat{\mathfrak{g}})} \otimes f_*)(T(z))$. Next applying $\text{Id}_{U_q(\widehat{\mathfrak{g}})} \otimes \tau_J \otimes \tau_J$ to the above equation,

$$\hat{T}_{12}(z) \hat{T}_{13}(w) (R_q^J(z, w)^{-1})_{23} = (R_q^J(z, w)^{-1})_{23} \hat{T}_{13}(w) \hat{T}_{12}(z).$$

Here $\hat{T}(z) = (\text{Id}_{U_q(\widehat{\mathfrak{g}})} \otimes \tau_J)(\tilde{T}(z))$. In other words,

$$R_{23}^J(z, w) \hat{T}_{12}(z) \hat{T}_{13}(w) = \hat{T}_{13}(w) \hat{T}_{12}(z) R_{23}^J(z, w) \in (U_q(\widehat{\mathfrak{g}}) \otimes \text{End} \mathbf{V}^J)((z^{-1}, w^{-1})).$$

The same argument applied to Relations (3.11)-(3.10), we get a well-defined superalgebra morphism: $f_{J,I} : U_q(\widehat{\mathfrak{g}}') \longrightarrow U_q(\widehat{\mathfrak{g}})$ such that

$$\begin{aligned} (f_{J,I} \otimes \text{Id}_{\text{End} \mathbf{V}^J})(S_J(z)) &= (\text{Id}_{U_q(\widehat{\mathfrak{g}})} \otimes \tau_J f_*)(S(z)), \\ (f_{J,I} \otimes \text{Id}_{\text{End} \mathbf{V}^J})(T_J(z)) &= (\text{Id}_{U_q(\widehat{\mathfrak{g}})} \otimes \tau_J f_*)(T(z)). \end{aligned}$$

The rest is now clear. \square

Remark 4.4. Let $M + 1 \leq r < M + N$. Let v be a highest ℓ -weight vector for the $U_q(\widehat{\mathfrak{g}}')$ -module $W_{k,a}^{(r)}$. Then $f_{J,I}^* v$ is a highest ℓ -weight vector in $f_{J,I}^* W_{k,a}^{(r)}$ with:

$$\begin{aligned} s_{ii,J}(z) f_{J,I}^* v &= f_{J,I}^* v \begin{cases} q^k - zaq^{-k} & (\text{if } i \leq N + M - r), \\ 1 - za & (\text{if } i > N + M - r), \end{cases} \\ t_{ii,J}(z) f_{J,I}^* v &= f_{J,I}^* v \begin{cases} q^{-k} - z^{-1}a^{-1}q^k & (\text{if } i \leq N + M - r), \\ 1 - za & (\text{if } i > N + M - r). \end{cases} \end{aligned}$$

In other words, $f_{J,I}^* W_{k,a}^{(r)} \cong W_{k,a;J}^{(N+M-r)}$ is a Kirillov-Reshetikhin module for the quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}}(N, M))$, corresponding to the fundamental weight ϖ_{N+M-r} . Clearly, $1 \leq N + M - r \leq N - 1$. Thus, to prove Theorem 4.2, we can assume $1 \leq r \leq M$.

4.3. A cyclicity result of Chari. To prove that a tensor product of $U_q(\widehat{\mathfrak{g}})$ -modules as in Theorem 4.2 is of highest ℓ -weight, it is enough to prove that a certain vector is generated by the highest ℓ -weight vector, as Chari did in the non-graded case [Ch02, Lemma 4.2].

Lemma 4.5. *Let V_+ (resp. V_-) be a $U_q(\widehat{\mathfrak{g}})$ -module of highest ℓ -weight (resp. of lowest ℓ -weight). Let $v_+ \in V_+$ (resp. $v_- \in V_-$) be a highest ℓ -weight vector (resp. lowest ℓ -weight vector). Then the $U_q(\widehat{\mathfrak{g}})$ -module $V_+ \otimes V_-$ (resp. $V_- \otimes V_+$) is generated by $v_+ \otimes v_-$ (resp. $v_- \otimes v_+$).*

Proof. If $v \in V$ is a highest/lowest ℓ -weight vector for a $U_q(\widehat{\mathfrak{g}})$ -module V , then according to Proposition 4.3, $f_{J,I}^* v$ is a highest/lowest ℓ -weight vector for the $U_q(\widehat{\mathfrak{g}}')$ -module $f_{J,I}^* V$. Hence, it is enough to prove the first part: $V_+ \otimes V_- = U_q(\widehat{\mathfrak{g}})(v_+ \otimes v_-)$.

As V_- is a lowest ℓ -weight $U_q(\widehat{\mathfrak{g}})$ -module with lowest ℓ -weight vector v_- , V_- is spanned as a vector superspace by the $X_{i_1, n_1}^+ X_{i_2, n_2}^+ \cdots X_{i_s, n_s}^+ v_-$ for $s \in \mathbb{Z}_{\geq 0}$ and $i_t \in I_0, n_t \in \mathbb{Z}$. In particular, with respect to the action of the $s_{ii}^{(0)}$, V_- is endowed with a $\mathbf{Q}_{\geq 0}$ -grading such that $(V_-)_\alpha$ is spanned by the above vectors with $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_s}$. This $\mathbf{Q}_{\geq 0}$ -grading in turn endows V_- with a $\mathbb{Z}_{\geq 0}$ -grading such that $(V_-)_n$ is spanned by the above vectors with $n = s$. We prove by induction on $n \in \mathbb{Z}_{\geq 0}$ that

$$(P_n) : V_+ \otimes (V_-)_n \subseteq U_q(\widehat{\mathfrak{g}})(v_+ \otimes v_-).$$

When $n = 0$, $(V_-)_0 = \mathbb{C}v_-$. For all $v \in V_+$, we have

$$X_{i,n}^-(v \otimes v_-) = X_{i,n}^- v \otimes v_-$$

since $(U_q(\widehat{\mathfrak{g}}))_{-\alpha} v_- = 0$ for $\alpha \in \mathbf{Q}_{\geq 0} \setminus \{0\}$. As V_+ is of highest ℓ -weight generated by the highest ℓ -weight vector v_+ , we get $V_+ \otimes v_- \subseteq U_q(\widehat{\mathfrak{g}})(v_+ \otimes v_-)$. Now assume (P_k) for $k \leq n$.

Let us prove (P_{n+1}) . Take \mathbb{Z}_2 -homogeneous vectors $v_1 \in V_+$ and $v_2 \in (V_-)_\beta \subseteq (V_-)_n$. We have

$$X_{i,n}^+(v_1 \otimes v_2) \in (-1)^{|i||v_1|} v_1 \otimes X_{i,n}^+ v_2 + \sum_{\alpha \in \mathbf{Q}_{\geq 0} \setminus \{0\}} (U_q(\widehat{\mathfrak{g}}))_\alpha v_1 \otimes (U_q(\widehat{\mathfrak{g}}))_{\alpha_i - \alpha} v_2.$$

On the other hand, for $\alpha \in \mathbf{Q}_{\geq 0} \setminus \{0\}$, by definition

$$(U_q(\widehat{\mathfrak{g}}))_{\alpha_i - \alpha} v_2 \subseteq (V_-)_{\beta + \alpha_i - \alpha} \subseteq \sum_{k \leq n} (V_-)_k.$$

It follows that $v_1 \otimes X_{i,n}^+ v_2 \in U_q(\widehat{\mathfrak{g}})(v_+ \otimes v_-)$. As $(V_-)_{n+1}$ is spanned by the $X_{i,n}^+ v_2$ with $v_2 \in (V_-)_n$, we conclude. \square

Our proof is slightly different from that of Chari [Ch02, Lemma 4.2] in the sense that we do not use the Drinfeld-Jimbo generators (see the end of §A.3).

4.4. Natural representations. From Lemma 3.2, Propositions 3.4 and 3.6 (PS4)-(PS5) together with Remark 3.7 (PS7) follows a representation $\rho_{(1)}$ of the quantum superalgebra $\mathcal{U}_q(\mathfrak{g})$ on \mathbf{V} :

$$(\rho_{(1)} \otimes \text{Id}_{\text{End} \mathbf{V}})(T) = (\text{Id}_{\text{End} \mathbf{V}} \otimes \tau)(R^{-1}), \quad (\rho_{(1)} \otimes \text{Id}_{\text{End} \mathbf{V}})(S) = (\text{Id}_{\text{End} \mathbf{V}} \otimes \tau)((R')^{-1}).$$

To be more precise,

$$\rho_{(1)}(s_{ii}) = q_i E_{ii} + \sum_{j \neq i} E_{jj} = \rho_{(1)}(t_{ii}^{-1}) \quad (\text{for } i \in I),$$

$$\rho_{(1)}(s_{ij}) = (q_i - q_i^{-1}) E_{ij}, \quad \rho_{(1)}(t_{ji}) = (q_i^{-1} - q_i) E_{ji} \quad (\text{for } 1 \leq i < j \leq M + N).$$

From Proposition 3.9 and Example 1 it follows that $\rho_{(0)} = \rho_{(1)} \circ DF$. In other words, the Ding-Frenkel isomorphism $DF : U_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$ respects the natural representations. We can therefore write $\mathbf{V} \cong L(\epsilon_1)$ as $\mathcal{U}_q(\mathfrak{g})$ -modules.

For $a \in \mathbb{C}^\times$, define $\rho_a := \rho_{(1)} \circ \text{ev}_a$. The representations (\mathbf{V}, ρ_a) are called *natural representations* of the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$. For simplicity, let $\mathbf{V}(a)$ be the $U_q(\widehat{\mathfrak{g}})$ -module corresponding to (\mathbf{V}, ρ_a) . It is clear that $\mathbf{V}(a) \cong W_{1,a}^{(1)}$ as $U_q(\widehat{\mathfrak{g}})$ -modules (assuming $M \neq 0$).

The following lemma says that Perk-Schultz R -matrices can be interpreted as intertwining operators, from which comes naturally the Yang-Baxter equation Proposition 3.6 (PS1).

Lemma 4.6. *Let $a, b \in \mathbb{C}^\times$. Then $c_{\mathbf{V}, \mathbf{V}} \circ R(z, w)|_{(z,w)=(a,b)} : \mathbf{V}(a) \otimes \mathbf{V}(b) \rightarrow \mathbf{V}(b) \otimes \mathbf{V}(a)$ is a morphism of $U_q(\widehat{\mathfrak{g}})$ -modules.* \square

The proof is direct, using properties of the Perk-Schultz R -matrix in Proposition 3.6. We shall not use this result in the sequel.

For natural representations, it is possible to determine completely the cyclicity condition.

Proposition 4.7. *Let $k \in \mathbb{Z}_{>0}$ and $a_i \in \mathbb{C}^\times$ for $1 \leq i \leq k$. The $U_q(\widehat{\mathfrak{g}})$ -module $\bigotimes_{i=1}^k \mathbf{V}(a_i)$ is of highest ℓ -weight if and only if $a_i \neq a_j q_1^{-2}$ for $1 \leq i < j \leq k$. It is of lowest ℓ -weight if and only if $a_i \neq a_j q_{M+N}^{-2}$ for $1 \leq i < j \leq k$.*

The proof of this proposition is postponed to §6.2.

5. REPRESENTATIONS OF THE q -YANGIAN $Y_q(\mathfrak{gl}(1, 1))$

Fix $M = N = 1$ and $\mathfrak{g} = \mathfrak{gl}(1, 1)$. We study the category \mathcal{F} finite-dimensional representations of the q -Yangian $Y_q(\mathfrak{g})$. Up to tensor product by one-dimensional modules, simple objects in \mathcal{F} are parametrized by rational functions as in [HJ12, Theorem 3.11]. Also, an explicit condition for a tensor product of simple objects to be of highest ℓ -weight is given in terms of poles and zeros of rational functions (Theorem 5.2).

5.1. Simple objects in \mathcal{F} . Let us first construct some obvious $Y_q(\mathfrak{g})$ -modules.

5.1.1. One-dimensional $Y_q(\mathfrak{g})$ -modules. Let $D = \mathbb{C}v$ be an one-dimensional $Y_q(\mathfrak{g})$ -module. As v is a highest/lowest ℓ -weight vector, there exist $s \in \mathbb{Z}_2, a, b \in \mathbb{C}^\times$ and $f(z), g(z) \in 1 + z\mathbb{C}[[z]]$ such that

$$|v| = s, \quad s_{11}(z)v = af(z)v, \quad s_{22}(z)v = bg(z)v, \quad s_{12}(z)v = s_{21}(z)v = 0.$$

It follows from Theorem 3.12 that $X_{1,n}^+v = 0 = X_{1,n+1}^-v$. Henceforth $K_1^+(z)(K_2^+(z))^{-1}v \in \mathbb{C}^\times v$. In other words, $f(z) = g(z)$. In summary, there are three types of one-dimensional $Y_q(\mathfrak{g})$ -modules: $\mathbb{C}_s, \mathbb{C}_{(a,b)}, \mathbb{C}_f$ where $s \in \mathbb{Z}_2, (a, b) \in (\mathbb{C}^\times)^2$ and $f \in 1 + z\mathbb{C}[[z]]$. All one-dimensional $Y_q(\mathfrak{g})$ -modules factorize uniquely into tensor products $\mathbb{C}_s \otimes \mathbb{C}_{(a,b)} \otimes \mathbb{C}_f$.

5.1.2. Evaluation modules in \mathcal{F} . Following [Ts12], let us define $\dot{\mathcal{U}}_q(\mathfrak{g})$ to be the superalgebra generated by $\dot{s}_{ij}, \dot{t}_{ji}, \dot{s}_{ii}^{-1}$ for $1 \leq i \leq j \leq 2$, with \mathbb{Z}_2 -degrees and defining relations the same as those for $\mathcal{U}_q(\mathfrak{g})$ in §3.3.4 except the last relation which is replaced by

$$\dot{s}_{ii}\dot{s}_{ii}^{-1} = 1 = \dot{s}_{ii}^{-1}\dot{s}_{ii}.$$

In particular, the \dot{t}_{ii} are not required to be invertible. From the proof of Proposition 3.8, we see that there are well-defined evaluation morphisms ev_a for $a \in \mathbb{C}^\times$

$$\text{ev}_a : Y_q(\mathfrak{g}) \longrightarrow \dot{\mathcal{U}}_q(\mathfrak{g}), \quad s_{ij}(z) \mapsto \dot{s}_{ij} - zat_{ij}.$$

As usual, we understand that $\dot{s}_{ji} = 0 = \dot{t}_{ij}$ when $i < j$. Clearly $\dot{\mathcal{U}}_q(\mathfrak{g})$ is \mathbf{Q} -graded with respect to the conjugate actions of the \dot{s}_{ii} . Let us write down the defining relations of $\dot{\mathcal{U}}_q(\mathfrak{g})$:

$$\begin{aligned} |\dot{s}_{ii}|_{\mathbf{Q}} &= |\dot{t}_{ii}|_{\mathbf{Q}} = 0, & |\dot{s}_{ij}|_{\mathbf{Q}} &= \epsilon_i - \epsilon_j = -|\dot{t}_{ji}|_{\mathbf{Q}}, \\ \dot{s}_{12}^2 &= 0 = \dot{t}_{21}^2, & \dot{s}_{12}\dot{t}_{21} + \dot{t}_{21}\dot{s}_{12} &= (q - q^{-1})(\dot{t}_{11}\dot{s}_{22} - \dot{s}_{11}\dot{t}_{22}). \end{aligned}$$

From the above presentation of $\dot{\mathcal{U}}_q(\mathfrak{g})$ and from the evaluation morphisms, it is easy to build up explicit representations for $Y_q(\mathfrak{g})$.

Let $a \in \mathbb{C}^\times$. We shall define two evaluation representations ρ_a^\pm of $Y_q(\mathfrak{g})$ on the vector superspace $\mathbf{V} = \mathbb{C}v_1 \oplus \mathbb{C}v_2$. It is enough to give their generating matrices $[\rho_a^\pm] := (\rho_a^\pm(s_{ij}(z)))_{1 \leq i, j \leq 2}$ with respect to the standard basis (v_1, v_2) . More precisely,

$$\begin{aligned} [\rho_a^+] &= \begin{pmatrix} (1 - za)E_{11} + (q^{-1} - zaq)E_{22} & (q - q^{-1})E_{12} \\ -zaE_{21} & E_{11} + q^{-1}E_{22} \end{pmatrix}, \\ [\rho_a^-] &= \begin{pmatrix} E_{11} + q^{-1}E_{22} & (q^{-1} - q)E_{12} \\ -zaE_{21} & (1 - za)E_{11} + (q^{-1} - zaq)E_{22} \end{pmatrix}. \end{aligned}$$

Let $L_{1,a}^\pm$ be the $Y_q(\mathfrak{g})$ -modules associated with the representations ρ_a^\pm .

5.1.3. *Classification of simple objects in \mathcal{F} .* Finite-dimensional simple $Y_q(\mathfrak{g})$ -modules are classified in terms of highest ℓ -weights in the following way.

Lemma 5.1. (1) *A finite-dimensional simple $Y_q(\mathfrak{g})$ -module must be of highest ℓ -weight.*

(2) *Let S be a simple $Y_q(\mathfrak{g})$ -module generated by a highest ℓ -weight vector v with*

$$|v| = \bar{0}, \quad s_{ii}(z)v = f_i(z)v, \quad f_i(z) \in (\mathbb{C}[[z]])^\times \quad \text{for } i = 1, 2.$$

Then S is finite-dimensional if and only if $\frac{f_1(z)}{f_2(z)} = \frac{P(z)}{Q(z)}$ for some polynomials $P(z), Q(z) \in \mathbb{C}[z]$ with non-zero constant terms.

Proof. The proof of Part (1) is the same as that of [Zh13, Lemma 4.12], by considering the action of the $s_{ii}^{(0)}, (s_{ii}^{(0)})^{-1}$. For Part (2), “only if” comes from Theorem 3.12 and [HJ12, Lemma 3.9]. For the “if” part, write

$$P(z) = a \prod_{i=1}^m (1 - zc_i), \quad Q(z) = b \prod_{j=1}^n (1 - zd_j), \quad c_i, d_j, a, b \in \mathbb{C}^\times.$$

Then S is a sub-quotient of the tensor product

$$\left(\bigotimes_{i=1}^m L_{1,c_i}^+ \right) \otimes \left(\bigotimes_{j=1}^n L_{1,d_j}^- \right) \otimes \mathbb{C}_{(f_1(0), f_2(0))} \otimes \mathbb{C}_{f'},$$

where $f'(z) = f_1(z)f_1(0)^{-1} \prod_{i=1}^m (1 - zc_i)^{-1}$. As the $L_{1,a}^\pm$ are always two-dimensional, S must be finite-dimensional. \square

Let us define \mathbf{R} to be the subset of $(\mathbb{C}[[z]])^\times$ consisting of power series of the form $P(z)Q(z)^{-1}$ with $P(z), Q(z) \in 1 + z\mathbb{C}[z]$. Identically, \mathbf{R} is the set of rational functions $f(z) \in \mathbb{C}(z)$ such that $f(0) = 1$. Here we view a rational function as a meromorphic function $\mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$. For such f , let $V(f)$ be the simple $Y_q(\mathfrak{g})$ -module generated by a highest ℓ -weight vector v satisfying

$$|v| = \bar{0}, \quad s_{11}(z)v = f(z)v, \quad s_{22}(z)v = v.$$

For example, when $a \in \mathbb{C}^\times$,

$$V(1 - za) \cong L_{1,a}^+, \quad V\left(\frac{1}{1 - za}\right) \cong L_{1,a}^- \otimes \mathbb{C}_{\frac{1}{1 - za}}.$$

According to Lemma 5.1, $V(f)$ is finite-dimensional. Moreover, All finite-dimensional simple $Y_q(\mathfrak{g})$ -modules can be factorized uniquely into $V(f) \otimes D$ with D one-dimensional and $f \in \mathbf{R}$.

5.2. Tensor product of simple modules. For $f \in \mathbf{R}$, let $Z(f)$ (resp. $P(f)$) be the set of zeros (resp. poles) of the meromorphic function f . It is possible that $\infty \in Z(f) \cup P(f)$. The main result of this section can be stated as follows.

Theorem 5.2. *Let $f_1, f_2, \dots, f_s \in \mathbf{R}$. For $1 \leq i \leq s$, let v_i be a highest ℓ -weight vector in the simple $Y_q(\mathfrak{g})$ -module $V(f_i)$. Let $V := \bigotimes_{i=1}^s V(f_i)$ and $v := \bigotimes_{i=1}^s v_i \in V$. Then*

- (a) $V = Y_q(\mathfrak{g})v$ if and only if $P(f_i) \cap Z(f_j) = \emptyset$ for all $1 \leq i < j \leq s$;
- (b) $Y_q(\mathfrak{g})v$ is the unique simple sub- $Y_q(\mathfrak{g})$ -module of V if and only if $Z(f_i) \cap P(f_j) = \emptyset$ for all $1 \leq i < j \leq s$;
- (c) V is simple if and only if $P(f_i) \cap Z(f_j) = \emptyset$ for all $1 \leq i \neq j \leq s$.

Remark 5.3. (1) The theorem above can be viewed as a super analogue of [CP91, Theorems 3.4, 4.8] on classification and construction of finite-dimensional simple $U_q(\widehat{\mathfrak{sl}}_2)$ -modules in terms of Drinfeld polynomials. See [MY14, Theorem 4.6] for a closer statement involving rational functions instead of Drinfeld polynomials.

(2) Let $a_n \in \mathbb{C}^\times$ be given for $n \in \mathbb{Z}_{\geq 0}$. Suppose $a_n \neq a_m$ whenever $n \neq m$. Then for $n \in \mathbb{Z}_{>0}$ the tensor product of $Y_q(\mathfrak{g})$ -modules

$$W_n := \left(\bigotimes_{i=0}^{n-1} V\left(\frac{1 - za_{i+1}}{1 - za_i}\right) \right) \otimes V\left(\frac{1}{1 - za_n}\right)$$

is of highest ℓ -weight, and its simple quotient is isomorphic to $V(\frac{1}{1 - za_0})$. Hence given a finite-dimensional simple module S , we find infinitely many finite-dimensional highest ℓ -weight modules whose simple quotients are isomorphic to S , and the dimensions of these modules can be arbitrarily large. This gives a clue on the Weyl modules defined in [Zh13, §4.1] for the quantum loop superalgebra $U_q(L\mathfrak{sl}(M, N))$.

The proof of Theorem 5.2 will be given in §5.3.

5.2.1. *Factorization into prime simple modules.* Let $f \in \mathbf{R}$. Write $f(z) = \frac{N(z)}{D(z)}$ where

$$N(z) = \prod_{i=1}^s (1 - za_i), \quad D(z) = \prod_{i=1}^t (1 - zb_i)$$

such that $a_i, b_i \in \mathbb{C}^\times$ and $a_i \neq b_j$ for $1 \leq i \leq s, 1 \leq j \leq t$. Then

$$\begin{aligned} V(f) &\cong \bigotimes_{i=1}^s V\left(\frac{1 - za_i}{1 - zb_i}\right) \quad \text{if } s = t, \\ V(f) &\cong \left(\bigotimes_{i=1}^t V\left(\frac{1 - za_i}{1 - zb_i}\right) \right) \otimes \left(\bigotimes_{j=t+1}^s V(1 - za_j) \right) \quad \text{if } s > t, \\ V(f) &\cong \left(\bigotimes_{i=1}^s V\left(\frac{1 - za_i}{1 - zb_i}\right) \right) \otimes \left(\bigotimes_{j=s+1}^t V\left(\frac{1}{1 - zb_j}\right) \right) \quad \text{if } s < t. \end{aligned}$$

According to Theorem 5.2, these are factorizations of simple modules into prime simple modules. Here by a *prime* simple module we mean a simple module S which can not be written as $S_1 \otimes S_2$ with S_i being modules of dimension > 1 [HL10, §2.2].

5.2.2. *Constructions of prime simple modules.* We have seen in §5.1.2 the explicit formulas for $V(1 - za)$ and $V(\frac{1}{1 - za})$. There still remains the third kind of prime simple modules, namely $V(\frac{1 - za}{1 - zb})$ for $a, b \in \mathbb{C}^\times$ and $a \neq b$. Indeed, it is easy to check the following without using Theorem 5.2 (2): the tensor product of highest ℓ -weight vectors in $V(1 - za) \otimes V(\frac{1}{1 - zb})$ generates the unique simple sub- $Y_q(\mathfrak{g})$ -module, which is two-dimensional and isomorphic to $V(\frac{1 - za}{1 - zb})$. Let $\rho_{a,b}$ be the corresponding representation of $Y_q(\mathfrak{g})$ on \mathbf{V} . After some base

change the generating matrix becomes

$$[\rho_{a,b}] = \begin{pmatrix} \frac{1-za}{1-zb}E_{11} + \frac{q^{-1}-zaq}{1-zb}E_{22} & \frac{(q^{-1}-q)(b-a)}{1-zb}E_{12} \\ \frac{-z}{1-zb}E_{21} & E_{11} + \frac{q^{-1}-zbq}{1-zb}E_{22} \end{pmatrix}.$$

Remark that the matrix $[\rho_{a,b}]$ is well-defined even if $ab = 0$. In particular, for $a \in \mathbb{C}^\times$, $[\rho_{a,0}]$ (resp. $[\rho_{0,a}]$) is a generating matrix for the representation associated to $V(1-za)$ (resp. to $V(\frac{1}{1-za})$). Hence all the prime simple modules are built upon the vector superspace \mathbf{V} .

5.2.3. Duals of prime simple modules. Fix $a, b \in \mathbb{C}$ such that $a \neq b$. Let $\rho_{a,b}$ be the representation of $Y_q(\mathfrak{g})$ on \mathbf{V} as in the preceding paragraph. Let $[\rho_{a,b}]$ be its generating matrix with respect to the standard basis (v_1, v_2) . Since $Y_q(\mathfrak{g})$ is a Hopf superalgebra, there exists naturally a representation $\rho_{a,b}^*$ of $Y_q(\mathfrak{g})$ on \mathbf{V}^* defined by:

$$\rho_{a,b}^*(x) := (\rho_{a,b}(Sx))^* \quad \text{for } x \in Y_q(\mathfrak{g}).$$

Here we adopt the notations in the proof of Lemma 3.3. Let (v_1^*, v_2^*) be the dual basis of \mathbf{V}^* with respect to (v_1, v_2) . Let $e_{ij} \in \text{End } \mathbf{V}^*$ be such that $e_{ij}v_k^* = \delta_{jk}v_i^*$. Then

$$E_{ii}^* = e_{ii}, \quad E_{12}^* = e_{21}, \quad E_{21}^* = -e_{12}.$$

Let us compute the generating matrix of $\rho_{a,b}^*$ with respect to the basis (v_2^*, v_1^*) . By definition, $[\rho_{a,b}^*]_{ij} = \rho_{a,b}(S(s_{ij}(z)))^*$. On the other hand, in view of Equation (3.16),

$$[\rho_{a,b}(S(s_{ij}(z)))]_{1 \leq i, j \leq 2} = [\rho_{a,b}]^{-1}$$

The matrices above should be seen as matrices over the superalgebra $\text{End } \mathbf{V}$. A direct calculation indicates:

$$[\rho_{a,b}]^{-1} = \begin{pmatrix} \frac{q^{-1}-zbq}{q^{-1}-zaq}E_{11} + \frac{1-zb}{q^{-1}-zaq}E_{22} & \frac{(q^{-1}-q)(a-b)}{q^{-1}-zaq}E_{12} \\ \frac{z}{q^{-1}-zaq}E_{21} & E_{11} + \frac{1-za}{q^{-1}-zaq}E_{22} \end{pmatrix},$$

from which we obtain the generating matrix of $\rho_{a,b}^*$ with respect to the basis (v_2^*, v_1^*) of \mathbf{V}^* :

$$[\rho_{a,b}^*] = \frac{1-za}{q^{-1}-zaq} \begin{pmatrix} \frac{1-zb}{1-za}e_{22} + \frac{q^{-1}-zbq}{1-za}e_{11} & \frac{(q^{-1}-q)(a-b)}{1-zb}e_{21} \\ \frac{-z}{1-za}e_{12} & e_{22} + \frac{q^{-1}-zaq}{1-za}e_{11} \end{pmatrix} \cong \frac{1-za}{q^{-1}-zaq} [\rho_{b,a}].$$

In the above equation, \cong means that the two matrices on both sides are of the same form. They are by no means in the same superalgebra. In conclusion, as $Y_q(\mathfrak{g})$ -modules:

$$V\left(\frac{1-za}{1-zb}\right)^* \cong \mathbb{C}_{\overline{1}} \otimes \mathbb{C}_{(q,q)} \otimes \mathbb{C}_{\frac{1-za}{1-zaq^2}} \otimes V\left(\frac{1-zb}{1-za}\right).$$

5.3. Proof of Theorem 5.2. Note that (c) follows directly from (a) and (b).

5.3.1. Tensor products of prime simple modules. Let us prove (a) and (b) under the condition that the $f_i \in \mathbf{R}$ are of the form $f_i(z) = \frac{1-za_i}{1-zb_i}$ where $a_i, b_i \in \mathbb{C}$ and $a_i \neq b_i$. In this case, $P(f_i) \cap Z(f_j) = \emptyset$ if and only if $b_i \neq a_j$. Moreover, the $V(f_i)$ are always two-dimensional, and $V(f_i)^* \cong V(f_i^{-1}) \otimes D_i$ for some one-dimensional module D_i . By definition of the dual modules, (b) is equivalent to the following statement:

- (b1) The tensor product $\bigotimes_{i=1}^s V(f_i)$ is of lowest ℓ -weight if and only if $a_i \neq b_j$ for $1 \leq i < j \leq s$.

Let us prove (a). For $1 \leq i \leq s$, let u_i^+ (resp. u_i^-) be a highest (resp. lowest) ℓ -weight vector in $V(f_i)$. Then from the explicit realization of $V(f_i)$ we see that

$$|u_i^+| = \bar{0}, \quad s_{11}(z)u_i^+ = \frac{1 - za_i}{1 - zb_i}u_i^+, \quad s_{22}(z)u_i^+ = u_i^+, \quad s_{21}(z)u_i^+ = \frac{z\lambda_i}{1 - zb_i}u_i^-$$

where $\lambda_i \in \mathbb{C}^\times$. We remark that Lemma 4.5 still holds when replacing $U_q(\widehat{\mathfrak{g}})$ -modules by $Y_q(\mathfrak{g})$ -modules. Indeed, if W is a highest ℓ -weight $Y_q(\mathfrak{g})$ -module with w a highest ℓ -weight vector, then by Theorem 3.12 we see that W is spanned by vectors of the form $X_{1,n_1}^- \cdots X_{1,n_r}^- v$ where $r \in \mathbb{Z}_{\geq 0}$ and $n_i \in \mathbb{Z}_{\geq 1}$ for $1 \leq i \leq r$. Hence the proof of Lemma 4.5 goes perfectly for $Y_q(\mathfrak{g})$ -modules.

Let $V := \bigotimes_{i=1}^s V(f_i)$ and $u := \bigotimes_{i=1}^s u_i^+$. Via the action of the $s_{ii}^{(0)}$, V and the $V(f_i)$ are \mathbf{Q} -graded:

$$(V)_\lambda := \{x \in V \mid s_{ii}^{(0)} x = q^{(\epsilon_i, \lambda)} x \text{ for } i = 1, 2\}.$$

As $|u_i^+|_{\mathbf{Q}} = 0, |u_i^-|_{\mathbf{Q}} = -\alpha_1$, we see that: $|u|_{\mathbf{Q}} = 0$; $(V)_\lambda \neq 0$ if and only if $\lambda = -t\alpha_1$ for some $0 \leq t \leq s$; $\dim(V)_{-t\alpha_1} = \binom{s}{t}$. In particular, $(V)_{-\alpha_1}$ is generated by the vectors

$$w_j := \left(\bigotimes_{i=1}^{j-1} u_i^+ \right) \otimes u_i^- \otimes \left(\bigotimes_{j=i+1}^s u_j^+ \right) \text{ for } 1 \leq j \leq s.$$

On the other hand, set $V' := Y_q(\mathfrak{g})u$. As a highest ℓ -weight module, V' is \mathbf{Q} -homogeneous. Moreover, from Theorem 3.12 we see that

$$(V')_{-\alpha_1} = \sum_{n \in \mathbb{Z}_{\geq 1}} \mathbb{C} X_{1,n}^- u = \sum_{n \in \mathbb{Z}_{\geq 1}} \mathbb{C} s_{21}^{(n)} u.$$

In other words, $(V')_{-\alpha_1}$ is generated by the coefficients of $s_{21}(z)u \in V[[z]]$.

Suppose first that $V = V'$ is of highest ℓ -weight. Then the coefficients of $s_{21}(z)u$ generate an s -dimensional subspace.

$$\begin{aligned} s_{21}(z)u &= \sum_{i=1}^s \left(\prod_{j=1}^{i-1} s_{22}(z)u_j^+ \right) \otimes s_{21}(z)u_i^+ \otimes \left(\bigotimes_{j=i+1}^s s_{11}(z)u_j^+ \right) \\ &= \sum_{i=1}^s \frac{z\lambda_i}{1 - zb_i} \prod_{j=i+1}^s \frac{1 - za_j}{1 - zb_j} \left(\bigotimes_{j=1}^{i-1} u_j^+ \right) \otimes u_i^- \otimes \left(\bigotimes_{j=i+1}^s u_j^+ \right) \\ &= \frac{z}{\prod_{l=1}^s (1 - zb_l)} \sum_{i=1}^s \lambda_i g_i(z) w_i. \end{aligned}$$

Here the $g_i(z) \in \mathbb{C}[z]$ are defined by

$$g_i(z) = \prod_{j=1}^{i-1} (1 - zb_j) \prod_{j=i+1}^s (1 - za_j).$$

It follows that the polynomials $g_i(z) \in \mathbb{C}[z]$ must be linearly independent. In view of Lemma 5.4 below, we must have $b_i \neq a_j$ for $1 \leq i < j \leq s$, as desired.

Next suppose that $b_i \neq a_j$ for $1 \leq i < j \leq s$. We show by induction on s that V is of highest ℓ -weight. For $s = 1$ this is evident. Assume $s > 1$. Then we can assume furthermore that $\bigotimes_{i=2}^s V(f_i)$ is of highest ℓ -weight. Now Lemma 4.5 says that

$$V = Y_q(\mathfrak{g})w_1.$$

Since $b_i \neq a_j$ for $1 \leq i < j \leq s$, the polynomials $g_i(z)$ are linearly independent (Lemma 5.4). Hence the coefficients of $s_{21}(z)u$ generate an s -dimensional subspace. It follows that $w_1 \in V'$. Hence $V = V'$ is of highest ℓ -weight.

Lemma 5.4. *Let $k \in \mathbb{Z}_{>0}$. Let $a_i, a'_i \in \mathbb{C}$ be given for $1 \leq i \leq k$. For $1 \leq j \leq k$, define*

$$f_j(z) := \left(\prod_{i=1}^{j-1} (1 - za_i) \right) \left(\prod_{i=j+1}^k (1 - za'_i) \right) \in \mathbb{C}[z].$$

Then the $f_j(z)$ are linearly independent if and only if $a_i \neq a'_j$ for all $1 \leq i < j \leq k$.

Proof. The k polynomials $f_j(z)$ are of degree $\leq k-1$. Introduce

$$f_1(z) \wedge f_2(z) \wedge \cdots \wedge f_k(z) = \Delta(1 \wedge z \wedge \cdots \wedge z^{k-1}) \in \wedge^k \mathbb{C}[z].$$

Then the $f_j(z)$ are linearly independent if and only if $\Delta \neq 0$. For $j+s \leq k$, take

$$f_j^{(s)}(z) = \left(\prod_{i=1}^{j-1} (1 - za_i) \right) \left(\prod_{i=j+s+1}^k (1 - za'_i) \right).$$

Then $f_j^{(0)}(z) = f_j(z)$ and

$$f_i^{(s)}(z) - f_{i+1}^{(s)}(z) = (a_i - a'_{i+s+1})zf_i^{(s+1)}(z)$$

for $i+s+1 \leq k$. Take $\omega = 1 \wedge z \wedge \cdots \wedge z^{k-1}$. We have

$$\begin{aligned} \Delta\omega &= \bigwedge_{i=1}^k f_i^{(0)}(z) = \left(\bigwedge_{i=1}^{k-1} f_i^{(0)}(z) - f_{i+1}^{(0)}(z) \right) \wedge f_k(z) = \left(\bigwedge_{i=1}^{k-1} (a_i - a'_{i+1})zf_i^{(1)}(z) \right) \wedge f_k^{(0)}(z) \\ &= \left(\prod_{i=1}^{k-1} (a_i - a'_{i+1}) \right) \left(\bigwedge_{i=1}^{k-1} zf_i^{(1)}(z) \right) \wedge 1 \\ &= \left(\prod_{i=1}^{k-1} (a'_{i+1} - a_i) \right) 1 \wedge \left(\bigwedge_{i=1}^{k-2} z(f_i^{(1)}(z) - f_{i+1}^{(1)}(z)) \right) \wedge zf_{k-1}^{(1)}(z) \\ &= \left(\prod_{i=1}^{k-1} (a'_{i+1} - a_i) \right) \left(\prod_{i=1}^{k-2} (a_i - a'_{i+2}) \right) 1 \wedge \left(\bigwedge_{i=1}^{k-2} z^2 f_i^{(2)}(z) \right) \wedge z \\ &= \left(\prod_{i=1}^{k-1} (a'_{i+1} - a_i) \right) \left(\prod_{i=1}^{k-2} (a'_{i+2} - a_i) \right) 1 \wedge z \wedge \left(\bigwedge_{i=1}^{k-2} z^2 f_i^{(2)}(z) \right) \\ &= \cdots = \prod_{1 \leq i < j \leq k} (a'_j - a_i) \omega. \end{aligned}$$

Clearly $\Delta \neq 0$ if and only if $a_i \neq a'_j$ for $1 \leq i < j \leq k$. □

This ends the proof of Theorem 5.2 (a) in the case where the $f_i(z)$ are of the form $\frac{1-za_i}{1-zb_i}$ with $a_i, b_i \in \mathbb{C}$ and $a_i \neq b_i$. Similar arguments lead to (b1) by considering lowest ℓ -weight vectors and by developing the series $s_{12}(z)(\bigotimes_{i=1}^s u_i^-) \in V[[z]]$.

5.3.2. End of proof. In general, given $f \in \mathbf{R}$, one can find a decomposition (not necessarily unique) $f = \prod_{i=1}^d f^{(i)}$ such that: $f^{(i)} = \frac{1-za_i}{1-zb_i}$ where $a_i, b_i \in \mathbb{C}$ and $a_i \neq b_i$; $a_i \neq b_j$ for $1 \leq i \neq j \leq d$. It follows that (§5.3.1)

$$V(f) \cong \bigotimes_{i=1}^d V(f^{(i)}), \quad P(f) = \bigcup_{i=1}^d P(f^{(i)}), \quad Z(f) = \bigcup_{i=1}^d Z(f^{(i)}).$$

Hence $V(f)^* \cong V(f^{-1}) \otimes D_f$ with D_f a one-dimensional module. (b) is equivalent to

(b2) The tensor product $\bigotimes_{i=1}^s V(f_i)$ is of lowest ℓ -weight if and only if $Z(f_i) \cap P(f_j) = \emptyset$ for all $1 \leq i < j \leq s$.

Now (a),(b2) follow easily from the factorization of simple modules and from the special case (§5.3.1) where the f_i are of the form $\frac{1-za_i}{1-zb_i}$. \square

As we see in Theorem 5.2, the conditions for a tensor product of finite-dimensional simple $Y_q(\mathfrak{g})$ -modules to be of highest ℓ -weight and to be of lowest ℓ -weight respectively are in general different, which is quite contrary to the non-graded case, where these two conditions are the same due to the Weyl group action.

6. PROOF OF THEOREM 4.2

The whole section is devoted to the proof of Theorem 4.2. The outline is as follows. In view of Remark 4.4, one can assume $1 \leq r \leq M$. In particular, $\widehat{M} > 0$. Next, by the following induction argument from $U_q(\widehat{\mathfrak{gl}(M, N)})$ -modules to $U_q(\widehat{\mathfrak{gl}(M, N+1)})$ -modules, we can assume furthermore $N > 0$. Then we shall prove the theorem by induction on r . For the initial step $r = 1$, Theorem 4.2 is a special case of Proposition 4.7.

Throughout the proof, we use the following convention. Let $f : A \rightarrow B$ be a morphism of superalgebras. Let V be a B -module. Suppose that W is a sub-vector-superspace of V stable under the action of $f(A)$. We write $f^\bullet W$ as the sub- A -module of f^*V induced by the action of $f(A)$ on W . (f^*W has no sense!)

6.1. Induction. Let $\mathfrak{g}' = \mathfrak{gl}(M, N)$ and $\mathfrak{g}'' = \mathfrak{gl}(M, N+1)$. Let $h : U_q(\widehat{\mathfrak{g}'}) \rightarrow U_q(\widehat{\mathfrak{g}''})$ be the superalgebra morphism defined by $s_{ij}(z) \mapsto s_{ij}(z), t_{ij}(z) \mapsto t_{ij}(z)$ for $1 \leq i, j \leq M+N$. Let $1 \leq r \leq M, k \in \mathbb{Z}_{>0}$ and $a_j \in \mathbb{C}^\times$ for $1 \leq j \leq k$. For $1 \leq j \leq k$, let v_j be a highest ℓ -weight vector in $\text{ev}_{a_j}^* L(\varpi_r; \mathfrak{g}'')$. Here we view ϖ_r as a weight associated to the Lie superalgebra \mathfrak{g}'' and $L(\varpi_r; \mathfrak{g}'')$ as a simple highest weight $\mathcal{U}_q(\mathfrak{g}'')$ -module of highest weight ϖ_r . Define

$$K(a_j) := h(U_q(\widehat{\mathfrak{g}'})v_j) \subseteq \text{ev}_a^* L(\varpi_r; \mathfrak{g}'').$$

Then $\bigotimes_{j=1}^k K(a_j)$ is a sub- $U_q(\widehat{\mathfrak{g}'})$ -module of $h^*(\bigotimes_{j=1}^k \text{ev}_{a_j}^* L(\varpi_r; \mathfrak{g}''))$. Moreover

$$h^\bullet(\bigotimes_{j=1}^k K(a_j)) \cong \bigotimes_{j=1}^k \text{ev}_{a_j}^* L(\varpi_r; \mathfrak{g}')$$

as $U_q(\widehat{\mathfrak{g}}')$ -modules. If the tensor product $\bigotimes_{j=1}^k \text{ev}_{a_j}^* L(\varpi_r; \mathfrak{g}'')$ of $U_q(\widehat{\mathfrak{g}}'')$ -modules is of highest ℓ -weight, then so is the corresponding tensor product of $U_q(\widehat{\mathfrak{g}}')$ -modules.

Assume in the rest of the section $N > 0$. Let U be the q -Yangian $Y_q(\mathfrak{gl}(1, 1))$ as in §5.

6.2. Proof of Proposition 4.7. The idea is the same as that of the proof of Theorem 5.2. As before, we prove only the highest ℓ -weight part.

We adopt the notations of §4.4. Let $V := \bigotimes_{j=1}^k \mathbf{V}(a_j)$. Let $v := v_1^{\otimes k} \in V$. As in §5.3.1, V is \mathbf{P} -graded via the action of the $s_{ii}^{(0)}$.

First suppose that V is of highest ℓ -weight. Then from Theorem 3.12 it follows that

$$(V)_{k\epsilon_1 - \alpha_1} = \sum_{n \geq 1} \mathbb{C} s_{21}^{(n)} v.$$

As in the proof of Theorem 5.2, we get an explicit expression of $s_{21}(z)v$, which implies that the following polynomials

$$f_j(z) = \left(\prod_{i=1}^{j-1} (1 - za_i) \right) \left(\prod_{i=j+1}^k (1 - za_i q^{-2}) \right)$$

for $1 \leq j \leq k$ are linearly independent. In view of Lemma 5.4, this says that $a_i \neq a_j q^{-2}$ for $1 \leq i < j \leq k$.

Next, assume that $a_i \neq a_j q^{-2}$ for $1 \leq i < j \leq k$. By induction on k , one can suppose that $\bigotimes_{i=2}^k \mathbf{V}(a_i)$ is of highest ℓ -weight. Note that $\mathbf{V}(a_1)$ is a lowest ℓ -weight $U_q(\widehat{\mathfrak{g}})$ -module with v_{M+N} a lowest ℓ -weight vector. It is enough to verify that (Lemma 4.5)

$$v_{M+N} \otimes v_1^{\otimes k-1} \in U_q(\widehat{\mathfrak{g}})v.$$

For $1 \leq j \leq k$, let $K(a_j)$ be the subspace of $\mathbf{V}(a_j)$ spanned by v_1, v_{M+N} . According to the ice rule Proposition 3.6 (PS3), there exists a morphism of superalgebras $f : U \rightarrow U_q(\widehat{\mathfrak{g}})$

$$s_{11}(z) \mapsto s_{11}(z), \quad s_{12}(z) \mapsto s_{1, M+N}(z), \quad s_{21}(z) \mapsto s_{M+N, 1}(z), \quad s_{22}(z) \mapsto s_{M+N, M+N}(z),$$

From weight gradings on $\mathbf{V}(a_j)$ and V , it follows that: the $K(a_j)$ are stable by U ; the tensor product $\bigotimes_{j=1}^k K(a_j)$ is stable by U ; as U -modules

$$f^\bullet \left(\bigotimes_{j=1}^k K(a_j) \right) \cong \bigotimes_{j=1}^k f^\bullet(K(a_j)).$$

Here the RHS should be understood as a tensor product of U -modules. From the explicit formula of the action of the $s_{ij}(z)$ on $\mathbf{V}(a)$ defined in §4.4, we see that

$$f^\bullet(K(a_j)) \cong \mathbb{C}_{(q, 1)} \otimes \mathbb{C}_{1 - za_j} \otimes V\left(\frac{1 - za_j q^{-2}}{1 - za_j}\right)$$

as U -modules. Now Theorem 5.2 (a) implies immediately that $\bigotimes_{j=1}^k f^\bullet(K(a_j))$ is of highest ℓ -weight. In particular,

$$v_{M+N} \otimes v_1^{\otimes k-1} \in U f^\bullet(v_1^{\otimes k}) \subseteq U_q(\widehat{\mathfrak{g}})v.$$

Hence, V is of highest ℓ -weight, as desired. \square

The initial step $r = 1$ for the induction argument on $1 \leq r \leq M$ has been established. Now suppose that $r > 1$. Let us consider the $U_q(\widehat{\mathfrak{g}})$ -module $W_{1,a}^{(r)}$.

6.3. Weight grading on $W_{1,a}^{(r)}$. Fix $a \in \mathbb{C}^\times$. The $U_q(\widehat{\mathfrak{g}})$ -module $W_{1,a}^{(r)} = \text{ev}_a^* L(\varpi_r)$ is \mathbf{P} -graded under the action of the $s_{ii}^{(0)}$. By Theorem 2.1, $(W_{1,a}^{(r)})_\lambda$ is non-zero if and only if

$$\lambda = \epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_r}$$

where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq M + N$ and $i_s < i_{s+1}$ if $i_s \leq M$. Moreover, for such λ , $(W_{1,a}^{(r)})_\lambda$ is always one-dimensional, and for $x \in (W_{1,a}^{(r)})_\lambda$,

$$s_{ii}(z)x = (q^{(\lambda, \epsilon_i)} - zaq^{-(\lambda, \epsilon_i)})x, \quad t_{ii}(z)x = (q^{-(\lambda, \epsilon_i)} - z^{-1}a^{-1}q^{(\lambda, \epsilon_i)})x.$$

Let v_a^+ (resp. v_a^-) be a highest (resp. lowest) ℓ -weight vector in $W_{1,a}^{(r)}$. Then

$$v_a^+ \in (W_{1,a}^{(r)})_{\varpi_r}, \quad v_a^- \in (W_{1,a}^{(r)})_{r\epsilon_{M+N}}.$$

Introduce $u_a^\pm \in W_{1,a}^{(r)}$

$$u_a^+ = s_{1,M+N}^{(0)} t_{M+N,1} v_a^-, \quad u_a^- = t_{M+N,1}^{(0)} v_a^+.$$

Then from the following Chevalley relation we get $u_a^\pm \neq 0$,

$$s_{1,M+N}^{(0)} t_{M+N,1}^{(0)} + t_{M+N,1}^{(0)} s_{1,M+N}^{(0)} = (q - q^{-1})(t_{11}^{(0)} s_{M+N,M+N}^{(0)} - s_{11}^{(0)} t_{M+N,M+N}^{(0)}).$$

Here we used the assumption that $N > 0$. In particular,

$$\mathbb{C}u_a^+ = (W_{1,a}^{(r)})_{\epsilon_1 + (r-1)\epsilon_{M+N}}, \quad \mathbb{C}u_a^- = (W_{1,a}^{(r)})_{\epsilon_2 + \cdots + \epsilon_r + \epsilon_{M+N}}.$$

Introduce vector subspaces $K^+(a) = \mathbb{C}v_a^+ + \mathbb{C}u_a^-$, $K^-(a) = \mathbb{C}v_a^- + \mathbb{C}u_a^+ \subseteq W_{1,a}^{(r)}$. The \mathbf{P} -grading on $W_{1,a}^{(r)}$ says that the subspaces $K^\pm(a)$ are both sub- U -module of $f^*W_{1,a}^{(r)}$. Let $f^\bullet K^\pm(a)$ be the U -modules thus obtained.

Claim. Let $k \in \mathbb{Z}_{>1}$ and $a_j \in \mathbb{C}^\times$ for $1 \leq j \leq k$. Then we have the following:

- (1) $\bigotimes_{j=2}^k K^+(a_j)$ is a sub- U -module of $f^*(\bigotimes_{j=2}^k W_{1,a_j}^{(r)})$;
- (2) $K^-(a_1) \otimes (\bigotimes_{j=2}^k K^+(a_j))$ is a sub- U -module of $f^*(\bigotimes_{j=1}^k W_{1,a_j}^{(r)})$;
- (3) as U -modules, there exists a canonical isomorphism

$$f^\bullet(K^-(a_1) \otimes (\bigotimes_{j=2}^k K^+(a_j))) \cong f^\bullet K^-(a_1) \otimes (\bigotimes_{j=2}^k f^\bullet K^+(a_j)).$$

The proof of the claim relies on the following facts:

- (4) if $s_{li}(z)K^+(a) \neq 0$ and $(i \in \{1, M+N\}, 1 < l < M+N)$, then $r < l < M+N$;
- (5) if $i \neq l$ and $r < l < M+N$, then $s_{il}(z)K^\pm(a) = 0$.

These are checked directly using the \mathbf{P} -grading on $W_{1,a}^{(r)}$.

Next, as U -modules, using notations in §5.1 we get

$$f^\bullet K^-(a) \cong \mathbb{C}_{(r-1)\bar{1}} \otimes \mathbb{C}_{(q, q^{1-r})} \otimes \mathbb{C}_{1-zaq^{2r-2}} \otimes V\left(\frac{1-zaq^{-2}}{1-zaq^{2r-2}}\right),$$

$$f^\bullet K^+(a) \cong \mathbb{C}_{(q,1)} \otimes \mathbb{C}_{\frac{1}{1-za}} \otimes V\left(\frac{1-zaq^{-2}}{1-za}\right).$$

6.4. End of proof. Let us be in the situation of Theorem 4.2 with $1 \leq r \leq M$. Write $a_j = aq^{x_j}$. We prove by induction on k that the tensor product $V = \bigotimes_{j=1}^k W_{1,a_j}^{(r)}$ is of highest ℓ -weight. Assume that the $U_q(\widehat{\mathfrak{g}})$ -module $\bigotimes_{j=2}^k W_{1,a_j}^{(r)}$ is of highest ℓ -weight. Then it is enough to ensure (Lemma 4.5)

$$x := v_{a_1}^- \otimes \left(\bigotimes_{j=2}^k v_{a_j}^+ \right) \in U_q(\widehat{\mathfrak{g}}) \left(\bigotimes_{j=1}^k v_{a_j}^+ \right).$$

Remark that by definition

$$x = v_{a_1}^- \otimes \left(\bigotimes_{j=2}^k v_{a_j}^+ \right) \in K^-(a_1) \otimes \left(\bigotimes_{j=2}^k K^+(a_j) \right) =: L_1.$$

The claim above says that L_1 is a sub- U -module of V . Moreover, as U -modules,

$$\begin{aligned} f^\bullet L_1 &\cong D \otimes V\left(\frac{1-za_1q^{-2}}{1-za_1q^{2r-2}}\right) \otimes \left(\bigotimes_{j=2}^k V\left(\frac{1-za_jq^{-2}}{1-za_j}\right) \right), \\ D &\cong \mathbb{C}_{(r-1)\overline{1}} \otimes \mathbb{C}_{(q^k, q^{1-r})} \otimes \mathbb{C}_{(1-za_1q^{2r-2}) \prod_{j=2}^k (1-za_j)}. \end{aligned}$$

The RHS of the first equation above is of highest ℓ -weight in view of Theorem 5.2 as $a_1q^{2r-2} \neq a_jq^{-2}$ for $2 \leq j \leq k$ and $a_j \neq a_lq^{-2}$ for $2 \leq j < l \leq k$. It follows that

$$x \in f(U)(u_{a_1}^+ \otimes \left(\bigotimes_{j=2}^k v_{a_j}^+ \right)) \subseteq U_q(\widehat{\mathfrak{g}})(u_{a_1}^+ \otimes \left(\bigotimes_{j=2}^k v_{a_j}^+ \right)).$$

We are left to verify in turn that

$$y := u_{a_1}^+ \otimes \left(\bigotimes_{j=2}^k v_{a_j}^+ \right) \in U_q(\widehat{\mathfrak{g}}) \left(\bigotimes_{j=1}^k v_{a_j}^+ \right).$$

Take U' to be the quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}}(\widehat{M-1}, N))$. From the ice rule, we get a superalgebra morphism $g : U' \longrightarrow U_q(\widehat{\mathfrak{g}})$ defined by

$$s_{ij}(z) \mapsto s_{i+1,j+1}(z), \quad t_{ij}(z) \mapsto t_{i+1,j+1}(z).$$

For $b \in \mathbb{C}^\times$, let $K(b) = g(U')v_b^+ \subseteq W_{1,b}^{(r)}$. Clearly, $K(b)$ is a sub- U' -module of $g^*W_{1,b}^{(r)}$. Moreover,

$$g^\bullet K(b) \cong \text{ev}_b^* L(\varpi_{r-1}; \mathfrak{gl}(M-1, N)), \quad u_b^+ \in K(b).$$

Now it is straightforward to verify: $\bigotimes_{j=1}^k K(a_j)$ is a sub- U' -module of $g^*(\bigotimes_{j=1}^k W_{1,a_j}^{(r)})$; there exist canonical isomorphisms of U' -modules

$$g^\bullet \left(\bigotimes_{j=1}^k K(a_j) \right) \cong \bigotimes_{j=1}^k g^\bullet K(a_j) \cong \bigotimes_{j=1}^k \text{ev}_{a_j}^* L(\varpi_{r-1}; \mathfrak{gl}(M-1, N)).$$

The induction hypothesis on r (which keeps N unchanged) shows that the RHS above is of highest ℓ -weight. Hence

$$y = u_{a_1}^+ \otimes \left(\bigotimes_{j=2}^k v_{a_j}^+ \right) \in g(U') \left(\bigotimes_{j=1}^k v_{a_j}^+ \right) \subseteq U_q(\widehat{\mathfrak{g}}) \left(\bigotimes_{j=1}^k v_{a_j}^+ \right).$$

This concludes the proof of Theorem 4.2. \square

Remark 6.1. Let $1 \leq r \leq M, k \in \mathbb{Z}_{>0}$ and $a_j \in \mathbb{C}^\times$ for $1 \leq j \leq k$. From the proof of Theorem 4.2 we see that the $U_q(\widehat{\mathfrak{g}})$ -module $\bigotimes_{j=1}^k W_{1,a_j}^{(r)}$ is of highest ℓ -weight provided that $\frac{a_i}{a_j} \notin \{q^{-2l} : 1 \leq l \leq r\}$ for all $1 \leq i < j \leq k$.

More general cyclicity results on tensor products of Kirillov-Reshetikhin modules of the form $\bigotimes_{j=1}^k W_{l_j, a_j}^{(r_j)}$ can be hopefully obtained in this way. For this purpose, it is necessary to determine first of all the zeros and poles of R -matrices between $W_{l_1, a_1}^{(r_1)}$ and $W_{l_2, a_2}^{(r_2)}$, in view of Kashiwara's cyclicity results in the non-graded case [Ka02]. In type A, this should be possible after a fusion procedure [DO94, Po13].

APPENDIX A. PROOF OF THE COPRODUCT FORMULAS

Proposition 3.13 is proved in essentially the same way as [Zh13, Prop.5.4]. However, it should be noted that the coproduct estimations in *loc. cit* are not enough as seen from the proof of Chari's Lemma 4.5.

Without loss of generality, we shall prove the coproduct formulas for $K_{j,s}^+, X_{i,n}^\pm$ for $i \in I_0, j \in I$ and $s, n \in \mathbb{Z}_{\geq 0}$. Proof of other cases is parallel.

For simplicity, let $\bar{U} := U_q(\widehat{\mathfrak{g}})$. From the Gauss decomposition in §3.4.1, we see that

$$X_{i,0}^- = t_{i+1,i}^{(0)} (t_{ii}^{(0)})^{-1}, \quad X_{1,1}^- = -s_{21}^{(1)} (s_{11}^{(0)})^{-1}, \quad K_{i,0}^+ = s_{ii}^{(0)} = (K_{i,0}^-)^{-1}.$$

In the following, for two vectors x, y in a vector space, we write $x \doteq y$ if $x \in \mathbb{C}^\times y$.

A.1. Quantum brackets. Let $x \in U_\alpha, y \in U_\beta$ be \mathbf{Q} -homogeneous. Define

$$[x, y] := xy - (-1)^{|\alpha||\beta|} q^{(\alpha, \beta)} yx.$$

Given $x_s \in U_{\beta_s}$ for $1 \leq s \leq r$, define *iterated quantum brackets*

$$[x_1, x_2, \dots, x_r]_L := [[x_1, x_2, \dots, x_{r-1}]_L, x_r], \quad [x_1, x_2, \dots, x_r]_R := [x_1, [x_2, \dots, x_{r-1}]_R].$$

Lemma A.1. $[X_{1,1}^-, X_{2,0}^-, X_{3,0}^-, \dots, X_{M+N-1,0}^-]_L \doteq s_{M+N,1}^{(1)} (s_{11}^{(0)})^{-1}$.

Proof. Fix $i, j, k \in I$ such that $i < j < k$. By taking the matrix coefficients of $v_j \otimes v_i \mapsto v_k \otimes v_j$ for the operator equation:

$$R_{23}(z, w) T_{12}(z) S_{13}(w) = S_{13}(w) T_{12}(z) R_{23}(z, w)$$

we see that

$$\begin{aligned} & (-1)^{|i|+|j|} (z-w) t_{kj}(z) s_{ji}(w) + z(q_i - q_i^{-1}) t_{jj}(z) s_{ki}(w) \\ &= (z-w) (-1)^{|i|+|j|} s_{ji}(w) t_{kj}(z) + w(q_i - q_i^{-1}) s_{jj}(w) t_{ki}(z). \end{aligned}$$

Next by comparing the coefficients of zw we get

$$(-1)^{|i|+|j|}(t_{kj}^{(0)}s_{ji}^{(1)} - s_{ji}^{(1)}t_{kj}^{(0)}) + (q_i - q_i^{-1})t_{jj}^{(0)}s_{ki}^{(1)} = 0.$$

In other words,

$$(A.29) \quad [s_{ji}^{(1)}, t_{kj}^{(0)}] = (q_j - q_j^{-1})t_{jj}^{(0)}s_{ki}^{(1)}.$$

Note that for $1 \leq j \leq M + N - 1$ we have

$$X_{1,1}^- = -s_{21}^{(1)}(s_{11}^{(0)})^{-1}, \quad X_{j,0}^- = t_{j+1,j}^{(0)}(t_{jj}^{(0)})^{-1}.$$

By repeatedly applying Equation (A.29) we find the desired quantum bracket. \square

A.2. Relations on Drinfeld generators. Let us introduce the $H_{i,s}$ for $i \in I$ and $s \in \mathbb{Z}_{>0}$ by the following functional equations:

$$K_i^+(z) = s_{ii}^{(0)} \exp((q_i - q_i^{-1}) \sum_{s \in \mathbb{Z}_{>0}} H_{i,s} z^s) \in U[[z]].$$

Clearly the $H_{i,s}$ commute with each other as the $K_{i,n}^+$ do. Moreover,

$$\begin{aligned} [X_{i,m}^+, X_{j,n}^+] &= \delta_{ij}(q_i - q_i^{-1})(\Psi_{i,m+n} - \delta_{m+n,0}s_{ii}^{(0)}(s_{i+1,i+1}^{(0)})^{-1}) \quad \text{for } i \in I_0, m+n \geq 0, \\ \sum_{k \geq 0} \Psi_{i,k} z^k &= (s_{ii}^{(0)})^{-1} s_{i+1,i+1}^{(0)} \exp((q - q^{-1}) \sum_{s \in \mathbb{Z}_{>0}} (d_{i+1}H_{i+1,s} - d_i H_{i,s}) z^s). \end{aligned}$$

Next from the relations between $K_i^+(z)$ and $X_j^\pm(w)$ we deduce that for $i \in I, j \in I_0$

$$\begin{aligned} [H_{i,s}, X_{j,n}^\pm] &= 0 \quad \text{if } i \neq j, j+1, \\ [H_{i,s}, X_{i,n}^\pm] &= \pm q_i^s \frac{[s]}{s} X_{i,n+s}^\pm, \\ [H_{i,s}, X_{i-1,n}^\pm] &= \mp q_i^{-s} \frac{[s]}{s} X_{i,n+s}^\pm. \end{aligned}$$

Set $h_{i,s} := d_i H_{i,s} - d_{i+1} H_{i+1,s}$ for $i \in I_0, s \in \mathbb{Z}_{>0}$. Then for $1 \leq i \leq M + N - 2$ one can find $c_{i+1} \in \mathbb{C}^\times$ such that

$$[h_{i,1}, X_{i+1,n}^\pm] = \pm c_{i+1} X_{i+1,n+1}^\pm.$$

Let $c_1 \in \mathbb{C}^\times$ be such that $[H_{1,1}, X_{1,n}^\pm] = \pm c_1 X_{1,n+1}^\pm$.

A.2.1. $h_{i,1}$ as quantum brackets. To distinguish with K which we have used before, let us introduce $L_i := s_{ii}^{(0)}(s_{i+1,i+1}^{(0)})^{-1}$ for $i \in I_0$. Introduce also

$$E_0 := s_{M+N,1}^{(1)}(s_{M+N,M+N}^{(0)})^{-1}, \quad E_i := X_{i,0}^+, \quad L_0 := (L_1 L_2 \cdots L_{M+N-1})^{-1}.$$

Then for $0 \leq i \leq M + N - 1$, we have

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes L_i^{-1}.$$

Lemma A.2. $h_{i,1} \doteq [E_i, E_{i-1}, E_{i-2}, \dots, E_1, E_{i+1}, E_{i+2}, \dots, E_{M+N-1}, E_0]_R$ for $i \in I_0$.

Proof. First, Let us first compute $h_{1,1}$. Note that

$$E_0 \doteq [X_{1,1}^-, X_{2,0}^-, X_{3,0}^-, \dots, X_{M+N-1,0}^-]_L L_1 L_2 \cdots L_{M+N-1}.$$

Now by induction on $2 \leq i \leq M+N-1$ it is easy to see that

$$[E_i, E_{i+1}, \dots, E_{M+N-1}, E_0]_R \doteq [X_{1,1}^-, X_{2,0}^-, \dots, X_{i-1,0}^-]_L L_1 L_2 \cdots L_{i-1}.$$

In particular, when $i = 2$ we obtain

$$[E_2, E_3, \dots, E_{M+N-1}, E_0]_R \doteq X_{1,1}^- L_1.$$

Using $[E_1, X_{1,1}^-] \doteq L_1^{-1} h_{1,1}$ we conclude that

$$[E_1, E_2, \dots, E_{M+N-1}]_R \doteq h_{1,1}.$$

Next, we have $[h_{1,1}, X_{2,0}^-] \doteq X_{2,1}^-$. In this way we compute $X_{2,1}^-$ as

$$\begin{aligned} X_{2,1}^- &\doteq [h_{1,1}, X_{2,0}^-] = [[E_1, E_2, \dots, E_{M+N-1}, E_0]_R, X_{2,0}^-] \\ &\doteq [E_1, [L_2 - L_2^{-1}, [E_3, \dots, E_{M+N-1}, E_0]_R]_{q^{-(\alpha_1, \alpha_1 + \alpha_2)}}]_{q^{-(\alpha_1, \alpha_1)}} \\ &\doteq [E_1, [E_3, \dots, E_{M+N-1}, E_0]_R L_2^{-1}]_{q^{-(\alpha_1, \alpha_1)}} \\ &\doteq [E_1, [E_3, \dots, E_{M+N-1}, E_0]_R]_{q^{-(\alpha_1, \alpha_1 + \alpha_2)}} L_2^{-1} = [E_1, E_3, \dots, E_{M+N-1}, E_0]_R L_2^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} h_{2,1} &\doteq [E_2, X_{2,1}^-] L_2 \doteq [E_2, [E_1, E_3, \dots, E_{M+N-1}, E_0]_R L_2^{-1}] L_2 \\ &\doteq [E_2, [E_1, E_3, \dots, E_{M+N-1}, E_0]_R]_{q^{-(\alpha_2, \alpha_2)}} = [E_2, E_1, E_3, \dots, E_{M+N-1}, E_0]_R. \end{aligned}$$

In general, by an induction argument on $2 \leq i \leq M+N-1$, one can show that

$$\begin{aligned} X_{i,1}^- &\doteq [E_{i-1}, E_{i-2}, \dots, E_1, E_{i+1}, E_{i+2}, \dots, E_{M+N-1}, E_0]_R L_i^{-1}, \\ h_{i,1} &\doteq [E_i, E_{i-1}, \dots, E_1, E_{i+1}, E_{i+2}, \dots, E_{M+N-1}, E_0]_R. \end{aligned}$$

This concludes the proof. \square

A.2.2. *Coproduct for $h_{i,1}$.* Introduce the length function $\ell : \mathbf{Q}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$ by

$$\ell\left(\sum_{i \in I_0} n_i \alpha_i\right) = \sum_{i \in I_0} n_i.$$

In the following, when we write $\ell(\alpha)$, it should be understood implicitly that $\alpha \in \mathbf{Q}_{\geq 0}$. For $i \in I_0$, let U_i be the subalgebra of U generated by the E_j with $j \in I_0 \setminus \{i\}$. Clearly U_i is a \mathbf{Q} -graded subalgebra.

Let us first consider $h_{1,1}$:

$$h_{1,1} \doteq [E_1, E_2, \dots, E_{M+N-1}, E_0]_R.$$

To compute $\Delta(h_{1,1})$, notice first that

$$[\Delta E_1, \Delta E_2, \dots, \Delta E_{M+N-1}, E_0 \otimes L_0^{-1}]_R = [E_1, E_2, \dots, E_{M+N-1}, E_0]_R \otimes 1.$$

Note that $|E_0|_Q = -(\alpha_1 + \alpha_2 + \dots + \alpha_{M+N-1})$. It follows that

$$\Delta(h_{1,1}) \in 1 \otimes h_{1,1} + h_{1,1} \otimes 1$$

$$\begin{aligned}
& + \sum_{i \in I_0} \mathbb{C}^\times [1 \otimes E_1, \otimes, 1 \otimes E_{i-1}, E_i \otimes L_i^{-1}, 1 \otimes E_{i+1}, \dots, 1 \otimes E_{M+N-1}, 1 \otimes E_0]_R \\
& + \sum_{\ell(\alpha) > 1} U_\alpha \otimes U_{-\alpha}.
\end{aligned}$$

By definition of quantum brackets, the middle term above (after summation) becomes

$$E_i \otimes [E_1, E_2, \dots, E_{i-1}, E_{i+1}, \dots, E_{M+N-1}, E_0]_R L_i^{-1} =: r_{1,i}.$$

In view of the proof of Lemma A.2, this is zero except $i = 1, 2$, and

$$\begin{aligned}
r_{1,1} & \doteq E_1 \otimes X_{1,1}^-, \\
r_{1,2} & \doteq E_2 \otimes [E_1, [X_{1,1}^-, X_{2,0}^-] L_1 L_2] L_2^{-1} \doteq E_2 \otimes X_{2,1}^-.
\end{aligned}$$

In other words, we have

$$\begin{aligned}
\Delta(h_{1,1}) & \in h_{1,1} \otimes 1 + 1 \otimes h_{1,1} + \sum_{i \in I_0: i \leq 2} \mathbb{C}^\times E_i \otimes X_{i,1}^- \\
& + \sum_{\ell(\alpha) > 1} (U_1)_\alpha \otimes U_{-\alpha} + \sum_{\ell(\alpha - \alpha_1) > 0} U_\alpha \otimes U_{-\alpha}.
\end{aligned}$$

Similar arguments applied to the $h_{i,1}$ lead to the following coproduct formulas.

Lemma A.3. *Let $1 \leq i \leq M + N - 1$. Then for all $s \in I_0$ we have*

$$\begin{aligned}
\Delta(h_{i,1}) & \in h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + \sum_{j \in I_0: i-1 \leq j \leq i+1} \mathbb{C}^\times E_j \otimes X_{j,1}^- \\
& + \sum_{\ell(\alpha) > 1} (U_s)_\alpha \otimes U_{-\alpha} + \sum_{\ell(\alpha - \alpha_s) > 0} U_\alpha \otimes U_{-\alpha}, \\
\Delta(H_{1,1}) & \in H_{1,1} \otimes 1 + 1 \otimes H_{1,1} + \mathbb{C}^\times E_1 \otimes X_{1,1}^- + \sum_{\ell(\alpha - \alpha_1) > 1} U_\alpha \otimes U_{-\alpha}.
\end{aligned}$$

A.2.3. Coproduct for $X_{i,n}^+$. Let A_i be the subalgebra of U generated by the $L_i^{-1}, \Psi_{i,n}$ for $n \in \mathbb{Z}_{\geq 0}$.

Lemma A.4. *For $i \in I_0$ and $n \in \mathbb{Z}_{\geq 0}$, we have*

$$(A.30) \quad \Delta(X_{i,n}^+) - 1 \otimes X_{i,n}^+ \in \sum_{m=0}^n X_{i,m}^+ \otimes A_i + \sum_{\ell(\alpha) > 1} (U_i)_\alpha \otimes U_{\alpha_i - \alpha} + \sum_{\ell(\alpha - \alpha_i) > 0} U_\alpha \otimes U_{\alpha_i - \alpha}.$$

Proof. Let us assume first $2 \leq i \leq M + N - 1$. Then $[h_{i-1,1}, X_{i,n}^+] = c_i X_{i,n+1}^+$ for $n \in \mathbb{Z}_{\geq 0}$. We prove the above coproduct formula by induction on $n \in \mathbb{Z}_{\geq 0}$. Clearly,

$$\Delta(X_{i,0}^+) = 1 \otimes X_{i,0}^+ + X_{i,0}^+ \otimes L_i^{-1} \in 1 \otimes X_{i,0}^+ + X_{i,0}^+ \otimes A_i.$$

Assume that the coproduct formula (A.30) is true for n . Remark that for $j \in I_0 \setminus \{i\}$

$$[E_j \otimes X_{j,1}^-, 1 \otimes X_{i,0}^+] = 0.$$

Lemma A.3 applied to $h_{i-1,1}$ with $s = i$ gives

$$\Delta(c_i X_{i,n+1}^+) \in c_i 1 \otimes X_{i,n+1}^+ + \mathbb{C}^\times E_i \otimes [X_{i,n}^+, X_{i,1}^-]$$

$$\begin{aligned}
& + \sum_{m=0}^n [h_{i-1,1}, X_{i,m}^+] \otimes A_i + \sum_{\ell(\alpha) > 1} (U_i)_\alpha \otimes U_{\alpha_i - \alpha} + \sum_{\ell(\alpha - \alpha_i) > 0} U_\alpha \otimes U_{\alpha_i - \alpha} \\
\subseteq & c_i 1 \otimes X_{i,n+1}^+ + \sum_{m=0}^{n+1} X_{i,m}^+ \otimes A_i \\
& + \sum_{\ell(\alpha) > 1} (U_i)_\alpha \otimes U_{\alpha_i - \alpha} + \sum_{\ell(\alpha - \alpha_i) > 0} U_\alpha \otimes U_{\alpha_i - \alpha}.
\end{aligned}$$

This establishes Equation (A.30).

Next, when $i = 1$, we use the relation $[H_{1,1}, X_{1,n}^\pm] = \pm c_1 X_{1,n+1}^\pm$ and the coproduct formula $\Delta(H_{1,1})$ in Lemma A.3. The rest is parallel as in the case $i > 1$. \square

Lemma A.4 can be viewed as a refinement of Equation (3.27). In a similar way, it is not difficult to prove Equation (3.28) by using Lemma A.3.

Corollary A.5. *For $i \in I_0$ and $n \in \mathbb{Z}_{\geq 0}$ we have*

$$(A.31) \quad \Delta(\Psi_{i,n}) \in A_i \otimes A_i + \sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha}.$$

Proof. For $n = 0$, this is clear since $\Psi_{i,0} = L_i$. For $n > 0$, we have

$$\Psi_{i,n} \doteq [X_{i,n}^+, X_{i,0}^-].$$

It is enough to consider the bracket $[\Delta(X_{i,n}^+), \Delta(X_{i,0}^-)]$. Remark that by definition

$$[X_{i,0}^-, x] = 0 \quad \text{for } x \in U_i.$$

Now Equation (A.31) follows from Equation (A.30) and from the fact that $\Delta(X_{i,0}^-) = L_i \otimes X_{i,0}^- + X_{i,0}^- \otimes 1$. \square

It is due to the proof of the above corollary that we introduce the subalgebras U_i .

A.3. Proof of Proposition 3.13. It is enough to prove Equation (3.26). Observe

$$\Delta(K_1^+(z)) = \Delta(s_{11}(z)) \in s_{11}(z) \otimes s_{11}(z) + \left(\sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha} \right) [[z]].$$

It is therefore enough to show that: for $i \in I_0$ and $n \in \mathbb{Z}_{\geq 0}$

$$\Delta(\Psi_{i,n}) \in \sum_{m=0}^n \Psi_{i,m} \otimes \Psi_{i,n-m} + \sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha}.$$

Clearly, $\Delta(\Psi_{i,0}) = \Psi_{i,0} \otimes \Psi_{i,0}$. In view of Corollary A.5 let us define $\Delta_i(\Psi_{i,n}) \in A_i \otimes A_i$ to be such that $\Delta(\Psi_{i,n}) - \Delta_i(\Psi_{i,n}) \in \sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha}$.

Fix $i \in I_0$. From the highest ℓ -weight representation theory (§4.1.1) of the quantum affine superalgebra U we observe that the subalgebra A_i is an algebra of Laurent polynomials:

$$A_i = \mathbb{C}[\Psi_{i,n} : n \in \mathbb{Z}_{>0}][\Psi_{i,0}, \Psi_{i,0}^{-1}].$$

So is the tensor algebra $A_i \otimes A_i$. It follows that an element $x \in A_i \otimes A_i$ is completely determined by the data $\chi \times \mu(x)$ where χ, μ are algebra homomorphisms $A_i \rightarrow \mathbb{C}$. Let us

fix $n \in \mathbb{Z}_{>0}$. Since $\sum_{\ell(\alpha)>0} U_\alpha \otimes U_{-\alpha}$ always kills the tensor product of two highest ℓ -weight vectors, we conclude that

$$\chi \times \mu(\Delta_i(\Psi_{i,n})) = \sum_{m=0}^n \chi(\Psi_{i,m}) \mu(\Psi_{i,n-m}) = \chi \times \mu\left(\sum_{m=0}^n \Psi_{i,m} \otimes \Psi_{i,n-m}\right).$$

It follows that $\Delta_i(\Psi_{i,n}) = \sum_{m=0}^n \Psi_{i,m} \otimes \Psi_{i,n-m}$. \square

To conclude this section, we remark that the elements $E_i, L_i^{\pm 1}$ with $0 \leq i \leq M + N - 1$ introduced in §A.2.1 satisfy all the relations of the Borel subalgebra of $U'_q(\widehat{\mathfrak{sl}(M, N)})$ defined by Drinfeld-Jimbo generators (See [Ya99, Proposition 6.7.1]). More excitingly, a q -character theory for finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ can be similarly developed as in [FR99], based on the above coproduct formulas.

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UNIVERSITÉ PARIS DIDEROT - PARIS 7, INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE
CNRS UMR 7586, BÂTIMENT SOPHIE GERMAIN, CASE 7012, 75025 PARIS CEDEX 13, FRANCE

E-mail address: huafeng.zhang@imj-prg.fr